University of Rostock
Institute of Communications Engineering (NT)

Design of RLC-Band Pass Filters

– Lecture script WS 2010/11 E.U.I.T.T. –

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The lecture treats the following topics:

Introduction, Filter design, Standard approximations

- Introduction and classification
- Analog Prototype Design
- Normalization, filter performance specifications, Standard filter type
- LC realization procedure, frequency transformation
- RLC-ladder networks circuits, denormalization of circuit and pole-zero map

Design of narrow band filter Description of compact filters

- N-circle coupled filter
- Resonant circuits, measures of coupling, alternate circuit
- Description of ladder networks circuits, chain fraction expansion
- Impedances, transfer functions

Circuit design

- Filter design, pole zero map
- Loss transformation
- Design of reactance four-pole networks
- Inductive and capacitive coupling
- Denormalizing

Computation of an example

- Definition of the design parameters
- Computation of the elements by MATLAB
- Simulation of the band filter circuit by PSpice
- Summary

References

[1] Saal, R.: Handbuch zum Filterentwurf
Hüthig, 1988

[2] G. C. Temes, S.K. Mitra: Modern Filter Theory and Design
John Wiley, New York, 1973

[3] A. I. Zverev: Handbook of Filter Synthesis
John Wiley, New York, 1967

[4] Herpy; Berka: Aktive RC-Filter
Budapest, 1984

[5] Kaufmann, F.:Synthese von Reaktanzfiltern
Oldenbourg, 1994

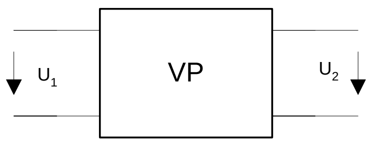
1 Introduction

Filters are technical realizations of given system functions, which affect the spectral characteristics of an input signal in the main (Frequency selection).
In the context of electro-technology the realizations with electrical networks interest as analog and digital circuits.
Filter applies for the separation of signal components with different frequency ranges e. g. in the telephone, radio communication etc. , to the influence of the signal spectrum e. g. by equalizer, sound controller etc. , to the suppression of disturbances, e. g. narrow band interference, noise.
Analog low pass and bandpass filter are needed as anti aliasing filter and interpolation filter in digital signal processing.

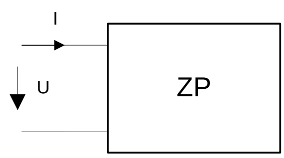
Filter	Electrotechnical realization of a given System function.
Condition	System function must be realizable by an electrical network as circuit.

System functions
System functions describe the input output behavior of a system. Filters are usually described by a system with one input and one output.

$F(s) = \frac{U_2(s)}{U_1(s)}$ Four-pole network function, Transfer function



$F(s) = \frac{U_1(s)}{I_1(s)}$ bzw. $\frac{I_1(s)}{U_1(s)}$ Two-pole network function, immitance function



If $F(s)$ is an rational fraction, the system is realizable by an electrical network.

$$F(s) = \frac{a_0 + a_1 \cdot s + \ldots + a_m \cdot s^m}{b_0 + b_1 \cdot s \ldots + b_n \cdot s^n}$$

$$F(s) = \frac{(s - z_1) \cdot (s - z_2) \cdot \ldots \cdot (s - z_m)}{(s - p_1) \cdot (s - p_2) \cdot \ldots \cdot (s - p_n)}$$

with:

m:	number of zeros
n:	number of poles, $n \geq m$
z_μ:	Zeros
p_ν:	Poles
$s = \sigma + j\omega$:	complex frequency

If $\sigma = 0$: $s \rightarrow j\omega \Rightarrow F(s) \rightarrow F(j\omega)$: Frequency response

The representation of the transfer function with the help of the pole zeros has the advantage in relation to the equivalent polynomial representation that the demanded numeric accuracy for the pole zeros can be smaller than those of the polynomial coefficients.

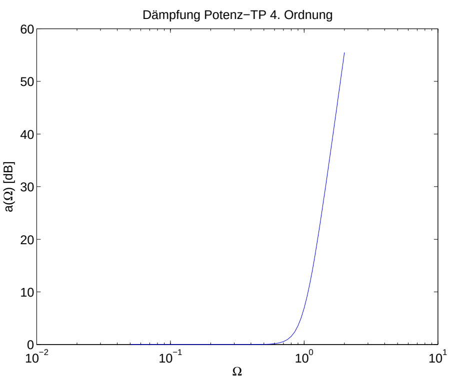
Derived frequency functions

The rational transfer function $F(j\omega)$ can be converted into the exponential form.

$$F(j\omega) = A(\omega) \cdot e^{jB(\omega)}$$

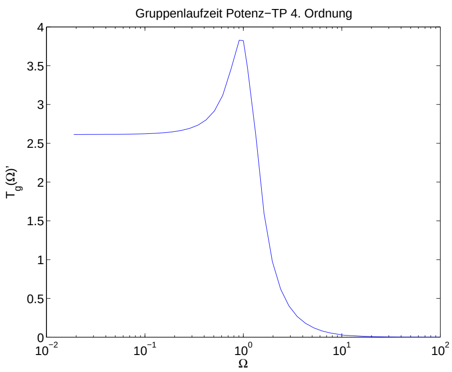
with:

$A(\omega) = \lvert F(j\omega)\rvert$:	Amplitude response
$B(\omega) = \arg F(j\omega)$:	Phase response
$a(\omega) = 20 \lg \frac{1}{A(\omega)}$:	Attenuation in dB
$T_g(\omega) = -\frac{dB(\omega)}{d\omega}$:	Group delay in s



$\Downarrow$

Filter



$\Downarrow$

All-pass filter , Delay sections, Impulse delay unit, Delay equalizer

The derived frequency functions can be used as specification for the design of the filters. The specification of the amplitude response and/or the attenuation are used to the design of so-called frequency filters. For the design of phase shifter networks (All-pass filter) phase response is specified and for the design of Delay equalizer the Group delay as specification is used.

On the other hand one can use the system functions for the filter analysis to evaluate he characteristics of the examined filter.

System functions in the time domain

The transfer function $F(s)$ can be convert by the inverse **Laplace**-Transformation into the time domain. The most important system functions in the time domain are:

$f(t) = \mathfrak{L}^{-1}\left\{F(s)\right\}$ Impulse response
Weighting function

$s(t) = \mathfrak{L}^{-1}\left\{\frac{1}{s} \cdot F(s)\right\}$ Step response
Transient response

Primarily system functions in the time domain are used for the filter analysis. For the specification of special filters e. g. matched filters however the impulse response are used.

Classification of frequency-selective filters

Basic types of frequency-selective filters

- Low pass filter LP
- High pass filter HP
- Band pass filter BP
- Band-stop filter BS
- All-pass filter AP

1.1 Steps of the filter synthesis

1. Approximation
   Approximation to a given curve of Attenuation bzw. Group delay by means of a allowable function, which is realizable as a circuit .
   Goal: PZ-Map of $F(s)$.

2. Realization
   Realization of the PZ-Maps of $F(s)$ by a circuit
   Possible circuits:

   a) Analogue circuits

      - RLC-circuits
      - RC-active circuits
      - SC-circuits (SC = switch-capacitor)

   b) Digital circuits

1.2 Normalization of electronic devices, time and frequency

Goal: Working which dimensionless values!

Bezugsgren R_B = Reference resistor

ω_B bzw. f_B = Reference frequency

Normalization of time and frequency:

$$\Omega = \frac{\omega}{\omega_B} = \frac{f}{f_B}$$

$$t' = t \cdot \omega_B$$

Normalization and Denormalization

Normalization	Denormalization
$r = \frac{R}{R_B}$	$R = r \cdot R_B$
$l = \frac{\omega_B \cdot L}{R_B}$	$L = l \cdot \frac{R_B}{\omega_B}$
$c = \omega_B \cdot C \cdot R_B$	$C = \frac{c}{\omega_B \cdot R_B}$

In the further only frequency filters are examined.

2 Filter design, approximation

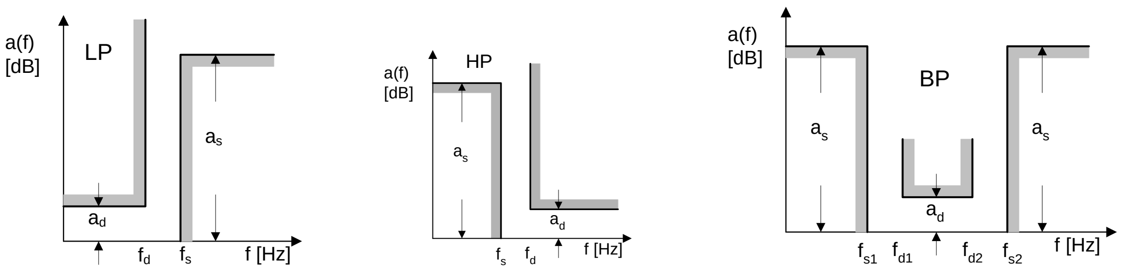
The task of the filtering design consists of sketching filters with a given damping trajectory. Because the system functions must meet boundary conditions, like realizability as circuit, the resulting system functions are only Approximation of the filter specifications.

Task: Design of frequency filters with given specifications in the basic forms Low-pass (LP), High-pass (HP), Band-pass (BP) and Band-stop (BS).

2.1 Standard approximations

Specified: Damping tolerance schema (DTS)

The Damping tolerance schema is in the simplest form a rectangular area for the passband or stopband with corresponding cutoff frequencies and limits of attenuation.



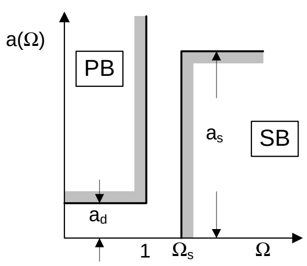
f_d = Pass band cut-off frequency in Hz
f_s = Stop band cut-off frequency in Hz
a_d = maximum pass band attenuation in dB
a_s = minimum stop band attenuation in dB

Condition for symmetrical BP or BS:
$f_{s1} \cdot f_{s2} = f_{d1} \cdot f_{d2} = f_m{}^2$
geometrical symmetrie
f_m = center frequency

Normalization of the Damping Tolerance Schema (DTS)

Goal: Uniform design procedure for lowpass (LP), high pass (HP) and symmetrical bandpass (BP) or band stop (BS) filters

One gets a normalized LP-Damping Tolerance Schema. The reference frequency of normalization is the pass band cut-off frequency f_d for LP and HP filters or the center frequency f_m of bandpass and band stop filters.



Normalization conditions:

LP $\rightarrow$ norm. LP: $\Omega = \frac{f}{f_d}$

HP $\rightarrow$ norm. LP: $\Omega = \frac{f_d}{f}$

symmetrical BP $\rightarrow$ norm. LP: $\Omega = \frac{f^2 - f_{d1} \cdot f_{d2}}{f \cdot (f_{d2} - f_{d1})}$

For the normalized LP-Damping Tolerance Schema the following relations result:

$\Omega =$ normalized frequency
$\Omega_s =$ normalized stop band cut-off frequency (normalized pass band cut-off frequency $\Omega_d = 1$)
$DB =$ Pass band area, characterized by a_d
$SB =$ Stop band area, characterized by a_s

Approximation of the normalized LP-Damping Tolerance Schema:

The aim of the approximation is to find parameters of suitable attenuation functions $a(\Omega)$.

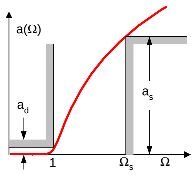
Conditions:

1. Fulfilment of the boundary conditions of the DTS.
2. $a(\Omega)$ must be an attenuation function of a realizable circuit

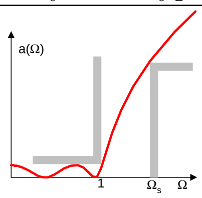
$\Rightarrow$ Standard approximations

Common Standard approximations:

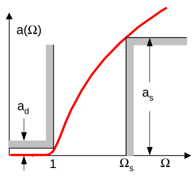
Monotonous slope of $a(\Omega)$
$\Rightarrow$ Butterworth filter (B-Filter)



Ripple in the passband
$\Rightarrow$ Chebyshev Type I filters (T1-Filter)



2.1.1 Butterworth (Power)-Approximation



Attenuation function

$$\boxed{a(\Omega) = 10 \cdot lg[1 + (\delta \cdot \Omega^n)^2]}$$

$\delta =$ Parameter
$n =$ integer number
$\delta \cdot \Omega^n = D(\Omega) =$ Power function

Calculation of δ and n for a given DTS:

For $\Omega = 1$: $a(1) = a_d = 10 \cdot lg(1+\delta^2)$

$$\Rightarrow \boxed{\delta = \sqrt{10^{0,1 \cdot a_d} - 1}}$$

If $a(\Omega) = a_B(\Omega) =$ Attenuation, than:
$\delta = D(\Omega) =$ Restriction

For $\Omega = \Omega_s$: $a(\Omega_s) = a_s = 10 \cdot lg\left[1 + (\delta \cdot {\Omega_s}^n)^2\right]$

$$\Rightarrow \boxed{n \geq \frac{lg\left[\left(\sqrt{10^{0,1 \cdot a_s} - 1}\right)/\delta\right]}{lg\Omega_s}}$$

$n =$ next integer number

Task: $a(\Omega) \rightarrow H(s)$ with $s = u + j\Omega$: normalized complex variable

Solution:

$$a(\Omega) = 20 \cdot lg\frac{1}{|H(j\Omega)|} = 10 \cdot lg\frac{1}{|H(j\Omega)|^2}$$

$$|H(j\Omega)|^2 = \frac{1}{1+\delta^2 \cdot \Omega^{2n}} = \frac{1}{1+\delta^2 \cdot \frac{(j\Omega)^{2n}}{(j)^{2n}}} = \frac{1}{1+\frac{\delta^2}{(-1)^n} \cdot (j\Omega)^{2n}} \qquad j\Omega \rightarrow s$$

$$|H(s)|^2 = H(s) \cdot H(-s) = \frac{1}{1+\frac{\delta^2}{(-1)^n} \cdot s^{2n}}$$

Poles of $H(s) \cdot H(-s)$: $\quad 1 + \frac{\delta^2}{(-1)^n} \cdot s^{2n} = 0 \;\Rightarrow\; s^{2n} = (-1)^{n+1} \cdot \frac{1}{\delta^2}$

With $-1 = e^{j\pi}$ result: $\quad s^{2n} = \frac{1}{\delta^2} \cdot e^{j\pi(n+1)} \;\Rightarrow\; s_{pi} = \sqrt[2n]{\frac{1}{\delta^2} \cdot e^{j\pi(n+1)}}$

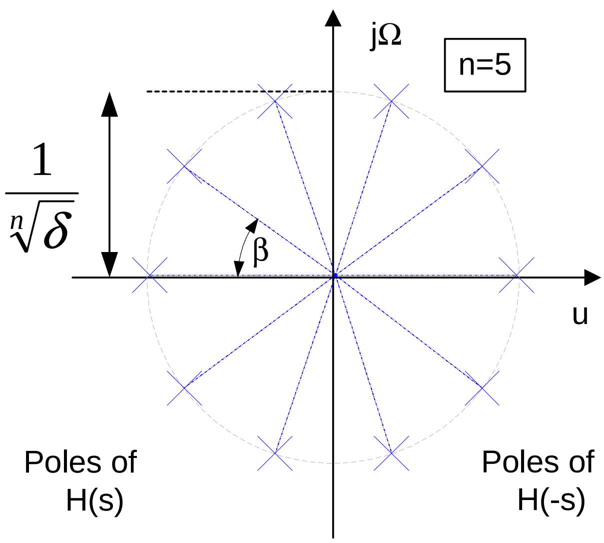
$\Rightarrow$ 2n - roots for p_{pi}

With the formula of Moivre: $\boxed{p_{\infty i} = \frac{1}{\sqrt[n]{\delta}} \cdot e^{\frac{j\pi(n+1+2i)}{2n}}}$ for i = 1, 2, ..., 2n

All poles are on a circle with the radius $1/\sqrt[n]{\delta}$.

The poles of the wanted transfer function $H(s)$ are the poles in the left complex half plane.

The poles enclose an angle of $\beta = \pi/n$.

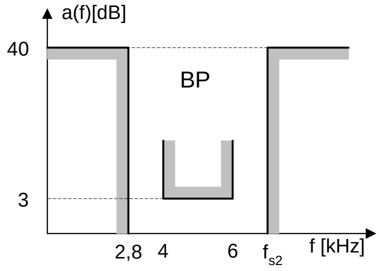


Example:

given: DTS of a symmetrical band-pass BP

wanted: PZ-Map of the normalized lowpass LP

required: Butterworth-Approximation



Solution:

stop band cut-off frequency: $$f_{s2} = \frac{f_{d1} \cdot f_{d2}}{f_{s1}} = \frac{4 \cdot 6}{2,8} = 8,57148\text{kHz}$$

Normalization of the DTS: $$\Omega_s = \frac{{f_{s2}}^2 - f_{d1} \cdot f_{d2}}{f_{s2}\left(f_{d2} - f_{d1}\right)} = \frac{8,57\ldots^2 - 4 \cdot 6}{8,57\ldots(6-4)} = \underline{2,8857142}$$

Determination of parameters:

$$\delta = \sqrt{10^{0,3} - 1} = 0,9976283 \cong \underline{1}$$

$$n \geq \frac{lg\left[\sqrt{10^4 - 1}/\delta\right]}{lg 2,88\ldots} = 4,3\cdots \Rightarrow n = 5$$

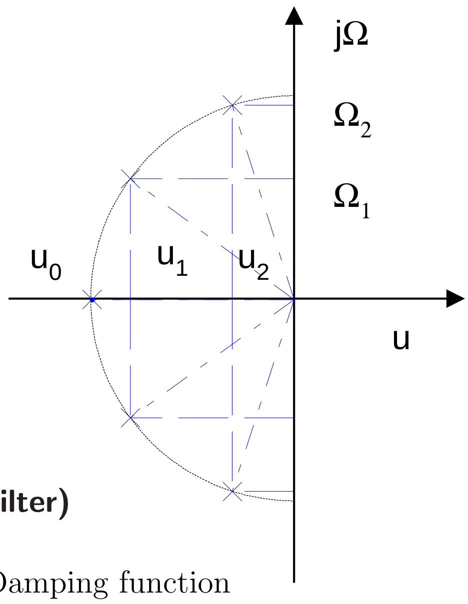
5 Poles on a semi-circle with the radius of unity.

$p_{\infty 0} = -u_0$

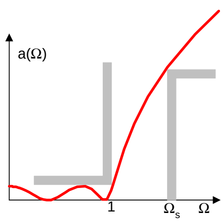
$p_{\infty 1,2} = -u_1 \pm j\Omega_1$

$p_{\infty 3,4} = -u_2 \pm j\Omega_2$

u_i	Ω_i
1	0
0,8090169	0,5877852
0,3090169	0,9510565



2.1.2 Chebyshev Type I Approximation (T1-Filter)



Damping function

$$\boxed{a(\Omega) = 10 \cdot \lg[1 + \delta^2 \cdot T_n(\Omega)^2]}$$

δ = Parameter

T_n = Chebyshev polynomial

Properties of Chebyshev polynomial:

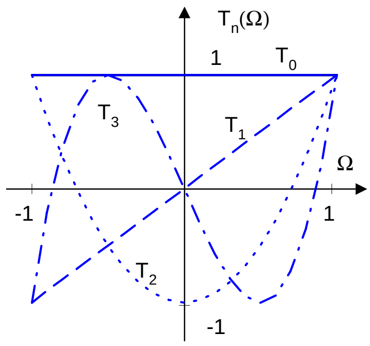
True for $\Omega \leq 1$:
(Pass band area)

$$\boxed{T_n(\Omega) = \cos\left(n \cdot \arccos \Omega\right)}$$

$T_0(\Omega) = 1$ $\qquad$ $T_3(\Omega) = 4 \cdot \Omega^3 - 3\Omega$

$T_1(\Omega) = \Omega$ $\qquad$ etc.

$T_1(\Omega) = 2 \cdot \Omega^2 - 1$



True for $\Omega > 1$:
(Stop band area)

$$\boxed{T_n(\Omega) = \cosh\left(n \cdot \text{arcosh}\Omega\right)}$$

monotone function with positive slope

Determination of δ and n for given DTS

For $\Omega = 1$: $\qquad a(1) = a_d = 10 \cdot \lg(1 + \delta^2) \;\Rightarrow\;$ $\boxed{\delta = \sqrt{10^{0,1 \cdot a_d} - 1}}$

just like the Butterworth-filter!

For $\Omega = \Omega_s$: $\qquad a(\Omega_s) = a_s = 10 \cdot \left[1 + \delta^2 \cdot \cosh^2\left(n \cdot \text{arcosh}\Omega_s\right)\right]$

$$\boxed{n \geq \frac{\text{arcosh}\left[\sqrt{10^{0,1 \cdot a_s} - 1}\,/\delta\right]}{\text{arcosh}\Omega_s}}$$

Determination of poles of $H(s)$

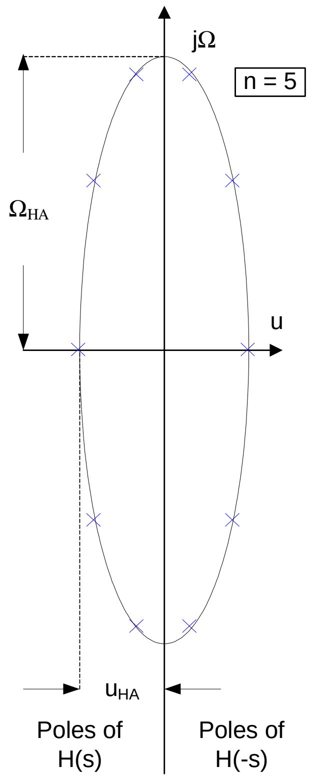
True for: $$H(s) \cdot H(-s) = \frac{1}{1 + \delta^2 \cdot T_n^2\left(\frac{s}{j}\right)}$$

Poles are the solution of: $$0 = 1 + \delta^2 \cdot T_n^2\left(\frac{s}{j}\right)$$

Result: Pole of T_1-Approximation are located at a ellipse.

Semi major and semi minor axes:

$$u_{HA} = \sinh\left(\frac{1}{n}\cdot \operatorname{arsinh}\frac{1}{\delta}\right)$$
$$\Omega_{HA} = \cosh\left(\frac{1}{n}\cdot \operatorname{arsinh}\frac{1}{\delta}\right)$$



Locations of poles: $p_k = u_k + j\Omega_k$

$$u_k = u_{HA}\cdot \sin\frac{\pi(2k-1)}{2n}$$
$$\Omega_k = \Omega_{HA}\cdot \cos\frac{\pi(2k-1)}{2n}$$

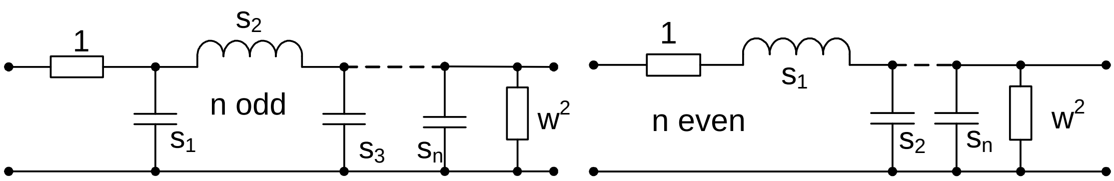
3 RLC-Filter-Realizations

3.1 RLC-ladder networks circuits

3.1.1 Closed solutions

Special solutions for P- and T1- Filter

given: δ, n



P-Filter

$$w^2 = 1, \qquad s_k = 2 \cdot \sqrt[n]{\delta} \cdot \sin \frac{(2k-1)\pi}{2n}$$

Test: $s_k = s_{n+1-k}$

with: $k = 1, 2, \ldots, n$

T1-Filter

$$w^2 = \begin{cases} 1 & \text{for } n \text{ odd} \\ \frac{\sqrt{1+\delta^2}+\delta}{\sqrt{1+\delta}-\delta} & \text{for } n \text{ even} \end{cases} \qquad s_k = \frac{a_k}{b_k}$$

with: $a_k = 2 \cdot \sin \frac{(2k-1) \cdot \pi}{2n}$

and: $b_k = \frac{1}{b_{k-1}} \left[b_0^2 + \left(sin \frac{2(k-1) \cdot \pi}{2n} \right)^2 \right]$

start value $b_0 = \sinh \left(\frac{1}{n} \operatorname{arsinh} \frac{1}{\delta} \right)$

3.2 Denormalizing

3.2.1 Denormalizing of RLC-Circuits

normalized devices		l	c	r
Target	Reference value	real electrical devices		
real LP	R_B $\omega_d = 2\pi f_d$	L $L = l \cdot \frac{R_B}{\omega_d}$	C $C = c\frac{1}{R_B \cdot \omega_d}$	
real HP	R_B $\omega_d = 2\pi f_d$	C $C = \frac{1}{l} \cdot \frac{1}{R_B \cdot \omega_d}$	L $L = \frac{1}{c} \cdot \frac{R_B}{\omega_d}$	R $R = r \cdot R_B$
real BP	R_B $\omega_{d1} = 2\pi f_{d1}$ $\omega_{d2} = 2\pi f_{d2}$	L C $L = l \cdot \frac{R_B}{\omega_{d2} - \omega_{d1}}$ $C = \frac{1}{l} \cdot \frac{\omega_{d2} - \omega_{d1}}{\omega_{d1} \cdot \omega_{d2} \cdot R_B}$	L $L = \frac{1}{c} \cdot \frac{(\omega_{d2} - \omega_{d1}) \cdot R_B}{\omega_{d2} \cdot \omega_{d1}}$ $C = c \cdot \frac{1}{(\omega_{d2} - \omega_{d1}) R_B}$	

3.2.2 Denormalizing of PZ-Maps

Given: PZ-Map of the normalized lowpass LP

wanted: PZ-data of the real TP, HP, BP etc.

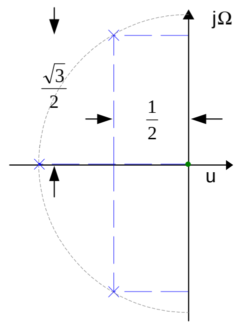
Transformation rules:

It is $s' = u + j\Omega =$ the complex variable of the normalized LP

LP	HP	BP
$s' \rightarrow \frac{s}{\omega_d}$	$s' \rightarrow \frac{\omega_d}{s}$	$s' \rightarrow \frac{1}{\Delta_{BP}} \left(\frac{\omega_m}{s} + \frac{s}{\omega_m} \right)$ $\Delta_{BP} = \frac{\omega_{d2} - \omega_{d1}}{\omega_m}; \omega_m = \sqrt{\omega_{d1} \cdot \omega_{d1}}$

Example for LP—BP - Transformation

$$H(s') = \frac{K}{(s'+1)\left(s'+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s'+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}$$



K = constant

Given: ω_m, Δ_{BP} of the BP
Transformation:

$$H(s) = \frac{K\cdot\Delta_{BP}\cdot\omega_m^3\cdot s^3}{\left[s^2+\omega_m\cdot\Delta_{BP}\cdot s+\omega_m^2\right]\left[s^2+\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\cdot\omega_m\cdot\Delta_{BP}\cdot s+\omega_m^2\right]\left[\ldots\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\ldots\right]}$$

with the poles:

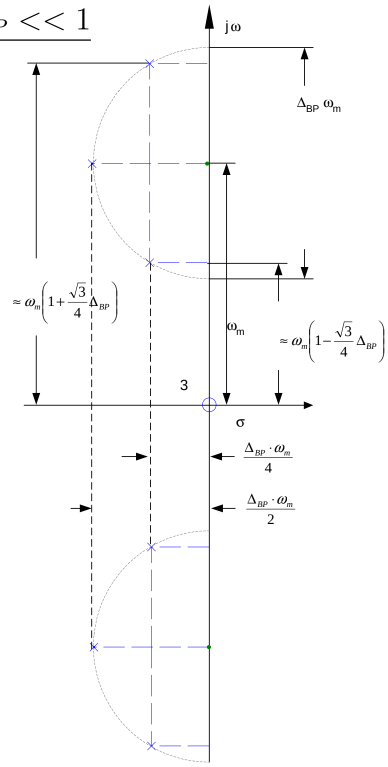
$$p_{1,2} = -\frac{\omega_m\cdot\Delta_{BP}}{2} \pm j\omega_m\sqrt{1-\left(\frac{\Delta_{BP}}{2}\right)^2}$$

$$p_{3,4} = -\frac{\omega_m\cdot\Delta_{BP}}{2}\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right) \pm j\omega_m\sqrt{1-\left[\frac{\Delta_{BP}}{2}\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\right]^2}$$

$$p_{5,6} = -\frac{\omega_m\cdot\Delta_{BP}}{2}\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right) \pm j\omega_m\sqrt{1-\left[\frac{\Delta_{BP}}{2}\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\right]^2}$$

The determination of the pole and zeros takes place with appropriate computational programs.

Approximation solutions for narrow band system: $\Delta_{BP} << 1$



$$p_{1,2} \approx -\frac{\omega_m \cdot \Delta_{BP}}{2} \pm j\omega_m$$
$$p_{3,4} \approx -\frac{\omega_m \cdot \Delta_{BP}}{4} \pm j\omega_m \left(1 - \frac{\sqrt{3}}{4} \cdot \Delta_{BP}\right)$$
$$p_{5,6} \approx -\frac{\omega_m \cdot \Delta_{BP}}{4} \pm j\omega_m \left(1 + \frac{\sqrt{3}}{4} \cdot \Delta_{BP}\right)$$

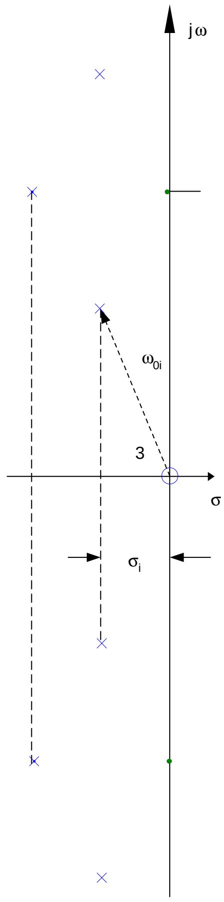
Generalization:

- Shift of the poles and zeros along the $j\omega$ - axis.
- Additional n zeros in the origin.

4 Design of narrow bandpass filters

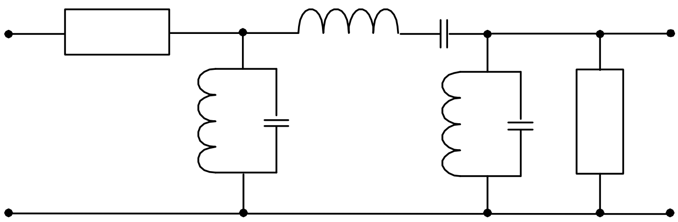
4.1 Problem:

HF band filters are narrow band systems
with $\Delta_{BP} << 1$



Realization possibilities:

1. Realization as reactance four-terminal network with source and load termination.



Requirement:

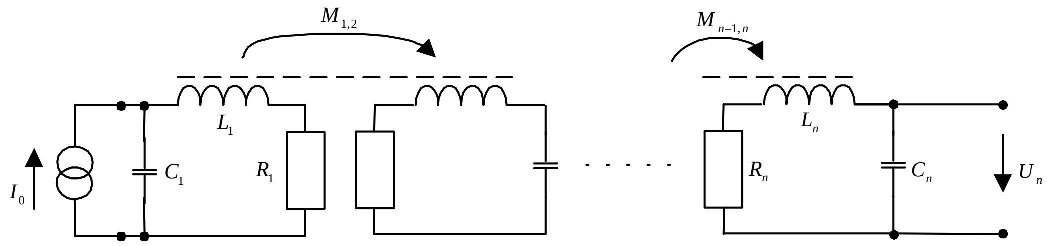
$$Q_L >> Q_p \text{ with: } Q_p = \frac{\omega_{0i}}{2\sigma_i} = \text{Pole quality factor}$$

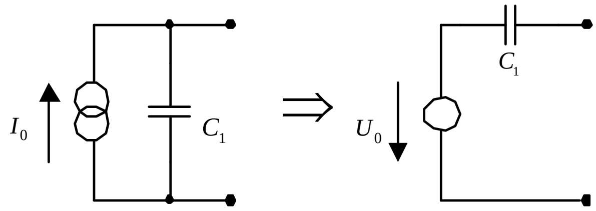
For HF circuits only with crystal filters reachable.
$\Longrightarrow \; Q \approx 10^4$.
Reactance condition for HF application heavily or not fulfillable.

2. HF filters with consideration of the losses in the elements:
   a) as multi-stage amplifiers with mutually detuned resonant circuits.
   b) Compact filter with coupled resonant circuits.

4.2 Description of the Compact filter with n resonant circuits

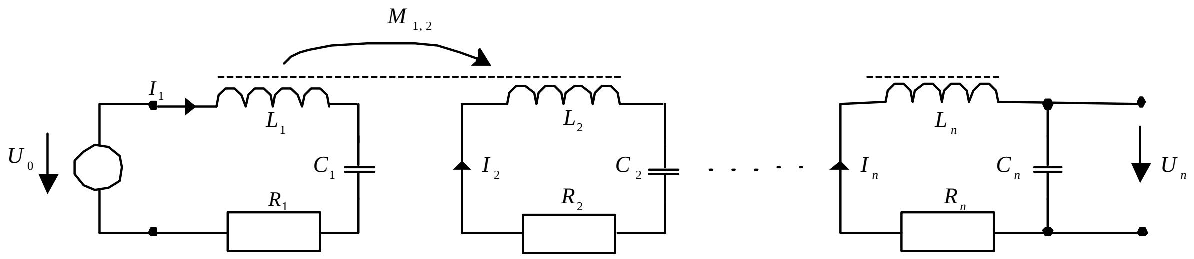
Assumption of an inductive coupling (other couplings later)
Circuit with lossy parallel resonant circuits.





The following relation applies to the conversion of a current source into a voltage source nearby the band center frequency:

$$U_0 \approx \frac{I_0}{j\omega_m \cdot C_1}$$



$M_{1,2}$ is the mutual inductance

Investigation of the individual circuit devices:

4.2.1 Resonant circuits

All resonant circuits are coordinated with the same resonant frequency!

$$Z = R + j\omega L + \frac{1}{j\omega C} = R\left[1 + jQ\left(\frac{\omega}{\omega_m} - \frac{\omega_m}{\omega}\right)\right]$$

with

$$Q = \frac{\omega_m L}{R} = \frac{1}{\omega_m C \cdot R} = \text{ Quality factor}$$
$$Z = R\left(1 + jQ \cdot V\right) \text{ with: } V = \frac{\omega}{\omega_m} - \frac{\omega_m}{\omega} \;=\; \underline{\text{Detuning}}$$

$$\text{Further parameters: } d = \frac{1}{Q} \; = \; \text{Damping factor}$$

$$B = \omega_{d2} - \omega_{d1} \; = \; \text{Bandwidth}$$

$$\text{Normalized Parameter: } \Delta = \frac{B}{\omega_m} \; = \; \text{Relative bandwidth}$$

$$\delta = \frac{d}{\Delta} \; = \; \frac{1}{Q \cdot \Delta} \; = \; \text{Normalized damping factor}$$

$$\Omega = \frac{V}{\Delta} \; = \; \text{Relative detuning, Normalized frequency}$$

With these parameters becomes:

$$Z = \frac{R}{\delta}\,(\delta + j\Omega) \;\Longrightarrow \text{Impedance function of the individual resonant circuit}$$

$$\text{change } j\Omega \;\rightarrow\; s = u + j\Omega \;\Longrightarrow \boxed{Z(s) = \frac{R}{\delta}\,(\delta + s)}$$

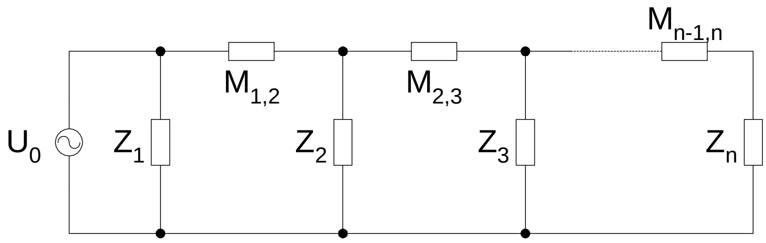
Normalized lowpass impedance function

4.2.2 Couple parameters

$$k_{ij} = \omega_m \cdot M_{ij} \qquad = \text{Coupling measure}$$

$$x_{ij} = k_{ij} \cdot \sqrt{\frac{\delta_i \cdot \delta_j}{R_i \cdot R_j}} = \begin{array}{l}\text{Coupling factor}\\ \text{(normalized coupling measure)}\end{array}$$

4.2.3 Total circuit



$$\begin{bmatrix} U_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{Z} \cdot \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix}$$

$$\text{mit:} \quad \vec{Z} = \begin{bmatrix} Z_1 & jk_{12} & 0 & \dots & 0 & 0 \\ jk_{12} & Z_2 & jk_{23} & & \vdots & \vdots \\ 0 & jk_{23} & Z_3 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & jk_{n-1,n} \\ 0 & 0 & 0 & \dots & jk_{n-1,n} & Z_n \end{bmatrix}$$

Mesh current matrix with consideration only the coupling of neighbour resonant circuits.

It is:

D = Determinant of the mesh current matrix

D_{11} = sub-determinant with cancellation of the 1. Line and 1. Column

Then applies:

$$Z_e = \frac{D}{D_{11}} = \text{input impedance of the BF}$$

Whereas:

$Z_e(s)$ = rational fractional function

$\Longrightarrow$ Continued fractions arrangement; Results in elements values

---

Continued fractions expansion: (first kind)

It is for example:

$$X(s) = \frac{s^4 + 10s^2 + 9}{s^3 + 4s} = \text{Function of a Reactance one port}$$

1. step:

$$(s^4 + 10s^2 + 9) : (s^3 + 4s) = s + \frac{6s^2 + 9}{s^3 + 4s} \; = \; s + \frac{1}{\frac{s^3+4s}{6s^2+9}}$$

$$s^4 + 4s^2$$

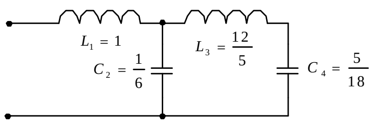
$$6s^2 + 9$$

further division in 2. step etc.

<u>Result:</u>

$$X(s) = s + \cfrac{1}{\frac{1}{6}s + \cfrac{1}{\frac{12}{5}s + \cfrac{1}{\frac{5}{18}s}}}$$

$$l_1 \quad c_2 \quad l_3 \quad c_4$$



If one divides the normalized input impedance of the total band filter circuit now into a continued fractions arrangement, one receives the following <u>result</u> after some simplifications:

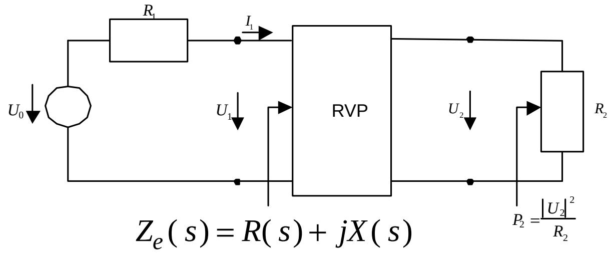
$$\frac{Z_e(s)}{R_1/\delta_1} = s + \delta_1 + \cfrac{x_{1,2}^2}{s + \delta_2 + \cfrac{x_{2,3}^2}{s + \delta_3 + \ddots \cfrac{x_{n-1,n}^2}{s + \delta_n}}}$$

The interesting at the continued fractions representation is that it represents a direct connection between rationally broken input impedance function and the dimensioning of the elements. Now still the problem exists to derive from a given transfer function the input impedance.

4.3 Determination of the input impedance

For the realization of a given transfer function (or PZ-Map) of the BP the relations between transfer function $H(s)$ and the input impedance $Z_e(s)$ must be intended.

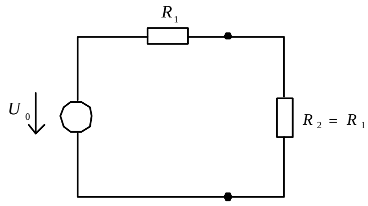
Starting point: Doubly-terminated two-port reactance network ⇒ In the four-terminal network no effective power is consumed.



It is valid:

$$P_2 = \frac{|U_2|^2}{R_2}$$

$$P_{max} = \frac{|U_0|^2}{4 \cdot R_1}$$



and with that:

$$\frac{P_2}{P_{max}} = \frac{|U_2|^2 \cdot 4 \cdot R_1}{|U_0|^2 \cdot R_2}$$

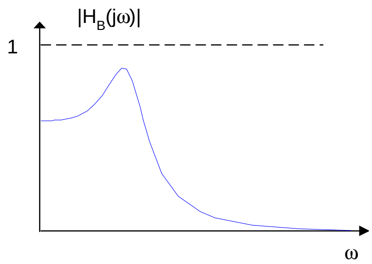
It then is:

$$|H_B| = \sqrt{\frac{P_2}{P_{max}}} = 2 \cdot \sqrt{\frac{R_1}{R_2}} \cdot \frac{|U_2|}{|U_0|}$$

One therefore defines:

$$H_B = 2 \cdot \sqrt{\frac{R_1}{R_2}} \cdot \frac{U_2(p)}{U_0(p)} \quad = \quad 2 \cdot \sqrt{\frac{R_1}{R_2}} \cdot H(p) \quad = \quad \underline{\text{Transmission function}}$$

Characteristics of $H_B(p)$:



$$|H_B|_{p=j\omega} = |H_B(j\omega)| \leq 1$$

Furthermore be valid:

$$\frac{U_1}{U_0} = \frac{Z_e}{R_1 + Z_e}$$

$$\left|\frac{U_1}{U_0}\right|^2 = \frac{|Z_e|^2}{|R_1 + Z_e|^2}$$

$$P_1 = \Re\{U_1 \cdot I_1^*\} \;=\; P_2$$

$$P_1 = \Re\left\{\frac{|U_1|^2}{Z_e^*}\right\} \;=\; \Re\left\{\frac{|U_1|^2(R + jX)}{R^2 + X^2}\right\}$$

$$P_1 = \frac{|U_1|^2 \cdot R}{|Z_e|^2} \;=\; P_2 = \frac{|U_2|^2}{R_2}$$

$$\Rightarrow \left|\frac{U_2}{U_1}\right|^2 \;=\; \frac{R_2 \cdot R}{|Z_e|^2}$$

$$\left|\frac{U_2}{U_0}\right|^2 = \left|\frac{U_1}{U_0}\right|^2 \cdot \left|\frac{U_2}{U_1}\right|^2 = \frac{R_2 \cdot R}{|R_1 + Z_e|^2}$$

From this, then follows:

$$|H_B|^2 = 4 \cdot \frac{R_1}{R_2} \cdot \left|\frac{U_2}{U_0}\right| \;=\; \frac{4 \cdot R_1 \cdot R}{|R_1 + Z_e|^2} \Rightarrow$$

Relation between transmission function and input impedance.

Reordering:

$$1 - |H_B|^2 = 1 - \frac{4 \cdot R_1 \cdot R}{|R_1 + Z_e|^2} \;=\; \frac{|R_1 + Z_e|^2 - 4 \cdot R_1 \cdot R}{|R_1 + Z_e|^2}$$

Applies to it:

$$\begin{aligned}|R_1 + Z_e|^2 - 4 \cdot R_1 \cdot R &= |R_1 + R + jX|^2 - 4 \cdot R_1 \cdot R = (R_1 + R)^2 + X^2 - 4 \cdot R_1 \cdot R \\ &= R_1^2 + R^2 + 2 \cdot R_1 \cdot R - 4 \cdot R_1 \cdot R + X^2 \;=\; (R_1 - R)^2 + X^2 \\ &= |R_1 - R - jX|^2 \;=\; |R_1 - Z_e|^2 \;=\; |Z_e - R_1|^2\end{aligned}$$

Becomes with that:

$$1 - |H_B|^2 = \left|\frac{R1 - Z_e(s)}{R1 + Z_e(s)}\right|^2 \;=\; \left|\frac{Z_e(s) - R1}{Z_e(s) + R1}\right|^2$$

One defines:

$$H_E(s) = \frac{Z_e(s) - R_1}{Z_e(s) + R_1} \;=$$

<u>Echo transmission coefficient</u> (Composite return current coefficient)

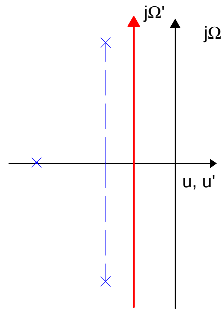
<u>Connections:</u>

$$\boxed{\begin{gathered}Z_e(s) = R_1 \cdot \frac{1 + H_E(s)}{1 - H_E(s)} \\ H_E(s) \cdot H_E(-s) = 1 - H_B(s) \cdot H_B(-s)\end{gathered}}$$

4.4 Circuit design

Idea of the procedure:

- Design method looked at till now permits only the realization by reactive four-terminal networks. (no losses!)
- Therefore loss transformation by moving of the $j\Omega$ - axis to the left.
  s-data $\Rightarrow$ s'-data
- Realization of a reactive four-terminal network for s'-data.
- The circuit then carries out the actual data of the s-plane with losses.



4.5 Loss transformation

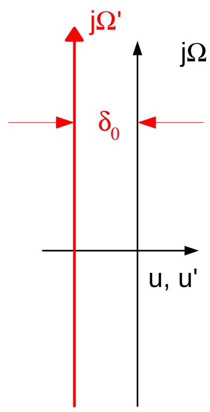
Formulation:

- All inner resonant circuits have the same quality factor and with that the same attenuation. δ_0
- The 1st circle has the attenuation $\delta 1 > \delta_0 \Rightarrow$ Consideration of the internal resistance of the source.
- The n'th circle has the attenuation $\delta_n > \delta_0 \Rightarrow$ Consideration of the input resistor of the following amplifier stage.

Loss transformation:

$$s + \delta_0 \;\Rightarrow\; s'$$

$$Z_e(s) \;\Rightarrow\; Z_e(s') \;\Rightarrow\; \text{Input impedance of a reactive four-terminal network } X_e(s')$$



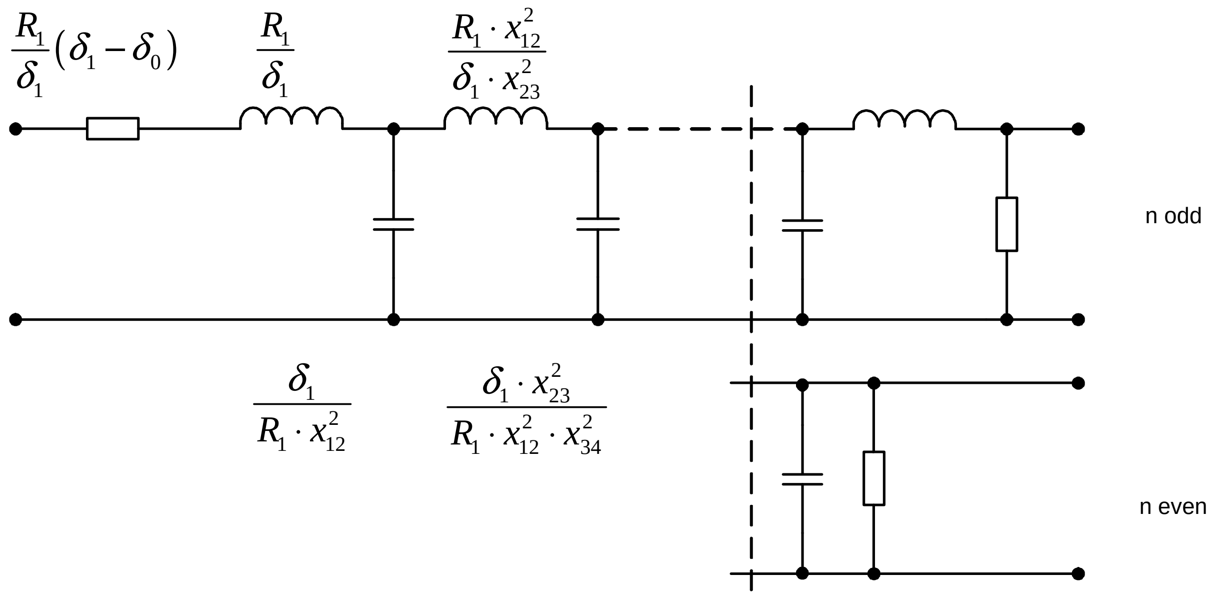
With that becomes the continued fraction decomposition of the input impedance for the coupled bandpass filters:

$$\frac{Z_e(s')}{R_1/\delta_1} = (\delta_1 - \delta_0) + s' + \cfrac{x_{1,2}^2}{s' + \cfrac{x_{2,3}^2}{s' + \ddots \cfrac{x_{n-1,n}^2}{(\delta_n - \delta_0) + s'}}}$$

Simplification:

$$Z_e(s') = \frac{R_1}{\delta_1}(\delta_1 - \delta_0) + s' \cdot \frac{R_1}{\delta_1} + \cfrac{1}{s' \cdot \frac{\delta_1}{R_1 \cdot x_{1,2}^2} + \cfrac{1}{s' \cdot \frac{R_1 \cdot x_{1,2}^2}{\delta_1 \cdot x_{2,3}^2} + \cfrac{1}{s' \cdot \frac{\delta_1 \cdot x_{2,3}^2}{R_1 \cdot x_{1,2}^2 \cdot x_{3,4}^2} + \ddots}}}$$

Resulting circuit:



The input impedance results from the loss transformation for a completed equivalent LP-Reactance from the input impedance of the doubly-terminated lossy bandpass circuit. The normalized components can be won directly by continued fraction decomposition of the fractional rational impedance function.
It is important to say that only transfer functions without zeros can be realized by coupled resonant circuits.

4.6 Further design method

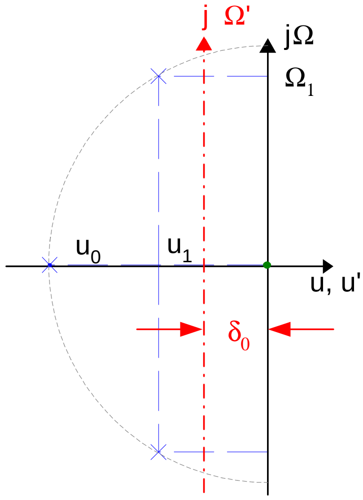
The further design method shall be shown at an example. The following steps are worked off:

1. Transformation of the given BP-DTS into the normalized LP-DTS and determination of the PZ-data.

2. Loss transformation with the aim of generating the PZ'-data of the s'-plan.
3. Building the transmission function from the PZ'-data and outline of the reactive four-terminal network.
4. Continued fraction stripping down of the input impedance for the determination of the normalized component values.
5. Denormalizing for the determination of the components of the coupled bandpass filter.

Classic example: Three section bandpass filter with Butterworth approximation.

Data: $f_m = 200$ kHz
B $= 4$ kHz
$a_d = 3$ dB



PZ-Data of the normalized LP:

u_k	Ω_k
1	0
0,5	0,8660254

Loss transformation:

Condition: $\delta_0 < \min\{|u_k|\} = 0{,}5$
Selected: $\delta_0 = 0{,}3$

Conclusions:

$$\delta_0 = \frac{1}{Q_0 \cdot \Delta} \quad \Rightarrow \quad Q_0 = \frac{1}{\delta_0 \cdot \Delta} = \frac{f_m}{\delta_0 \cdot B}$$

$$\underline{\underline{Q_0 = 166,\overline{6}}}$$

If the quality factor Q_0 (e.g. the filter coil) is predefined, then the bandwidth B cannot be chosen freely.

It is valid: $\boxed{B \geq \frac{f_m}{\delta_0 \cdot Q_0}}$

PZ data of the loss transformed LP:

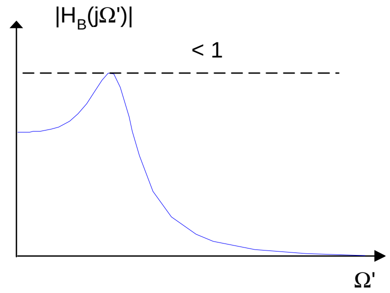
u'_k	Ω'_k
0,7	0
0,2	0,8660254

Transmission function:

$$H_B(s') = \frac{K}{(s'+0,7)(s'^2+0,4s'+0,2^2+0,8660254^2)}$$
$$= \frac{K}{s'^3+1,1s'^2+1,07s'+0,553}$$
$$H_B(s') = \frac{K}{N(s')}$$

K has to be chosen so, that

$$|H_B(s')|_{s'=j\Omega'} = |H_B(\Omega')| \quad \leq 1$$



Design of the reactive four-terminal network:

$$1 - H_B(s') \cdot H_B(-s') = H_E(s') \cdot H_E(-s')$$

$$H_E(s') \cdot H_E(-s') = \frac{N(s') \cdot N(-s') - K^2}{N(s') \cdot N(-s')} \quad \Longrightarrow$$

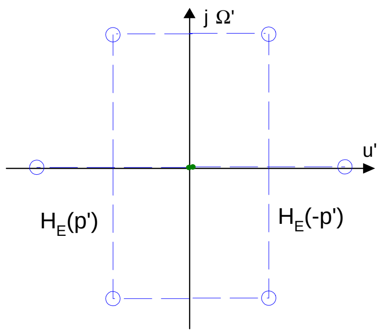
Determination of the zeros of $H_E(s') \cdot H_E(-s')$, since the poles are known.

$$H_E(s') \cdot H_E(-s') = \frac{-s'^6 - 0,93s'^4 + 0,0717s'^2 + 0,305809 - K^2}{-s'^6 - 0,93s'^4 + 0,0717s'^2 + 0,305809}$$

Approach: $K^2 = 0,1$

Zeros of $H_E(s') \cdot H_E(-s')$: $\Longrightarrow$ numerical solution!

u'_E	Ω'_E
+0,117077	+0,829324
+0,117077	-0,829324
-0,117077	+0,829324
-0,117077	-0,829324
+0,646687	0
-0,646687	0



Remark:
If K is set too greatly, so that $|H_E(j\Omega')| > 1$,the zeros of $H_E(s')$ and $H_E(-s')$ do not let themselves divide. The zeros then lie on the imaginary axis.

It gives up:

$$H_E(s') = \frac{s'^3 + 0,880841s'^2 + 0,8529096s' + 0,4536414}{s'^3 + 1,1s'^2 + 1,07s' + 0,553}$$

From this the input resistor $Z_e(s')$ is determined.

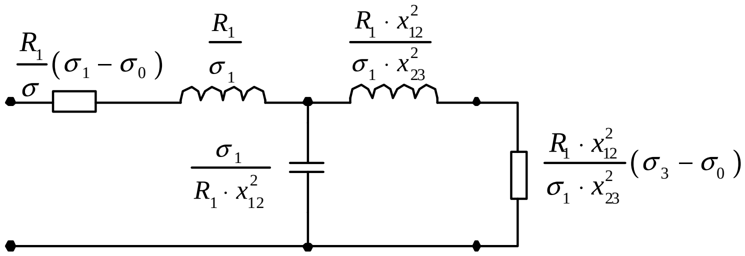
$$Z_e(s') = r_1 \cdot \frac{1 + H_E(s')}{1 - H_E(s')}$$

$$= \frac{2 \cdot s'^3 + 1,980841ss'^2 + 1,9229096s' + 1,0066414}{0,219159s'^2 + 0,2170904s' + 0,0993586} \cdot r_1$$

Continued fraction stripping down of $Z_e(s')/r_1$:

$$\frac{Z_e(s')}{r_1} = 1 + 9,1257945s' + \cfrac{1}{0,2156687s' + \cfrac{1}{10,227433s' + 10,131396}}$$

From this the normalized LP gives up:



The system of equations arises:

$$\frac{R_1}{\delta_1}(\delta_1 - \delta_0) = r_1 \tag{4.1}$$

$$\frac{R_1}{\delta_1} = 9,1257945 \cdot r_1 \tag{4.2}$$

$$\frac{\delta_1}{R_1 \cdot x_{1,2}^2} = 0,2156687/r_1 \tag{4.3}$$

$$\frac{R_1 \cdot x_{1,2}^2}{\delta_1 \cdot x_{2,3}^2} = 10,227433 \cdot r_1 \tag{4.4}$$

$$\frac{R_1 \cdot x_{1,2}^2}{\delta_1 \cdot x_{2,3}^2}(\delta_3 - \delta_0) = 10,131396 \cdot r_1 \tag{4.5}$$

One gets from (4.2):

$$\frac{R_1}{\delta_1 \cdot r_1} = 9,1257945$$

This expression is contained in all other equations.
$\Longrightarrow$ Recursive solution of the system of equations.

$$\boxed{\begin{array}{l} \text{Result:} \\ \begin{array}{rlcl} \delta_1 &= 0,4095795 & > & \delta_0! \\ x_{1,2}^2 &= 0,50809181 & \Rightarrow & x_{1,2} = 0,71280559 \\ x_{2,3}^2 &= 0,453363169 & \Rightarrow & x_{2,3} = 0,673322485 \\ \delta_3 &= 1,290609861 & > & \delta_0! \end{array} \end{array}}$$

Use of standard coils. $\Rightarrow$ Specification of an identical inductance for all circles.

Selected: $\underline{\underline{L \; = \; 0,1 \; mH}}$

Measure of the coupling:

It is valid:

$$x_{1,2}^2 = \omega_m^2 \cdot M_{1,2}^2 \cdot \frac{\delta_1 \cdot \delta_0}{R_1 \cdot R_0} = M_{1,2}^2 \cdot \frac{\omega_m^2}{B^2 \cdot L^2} \quad \Rightarrow$$

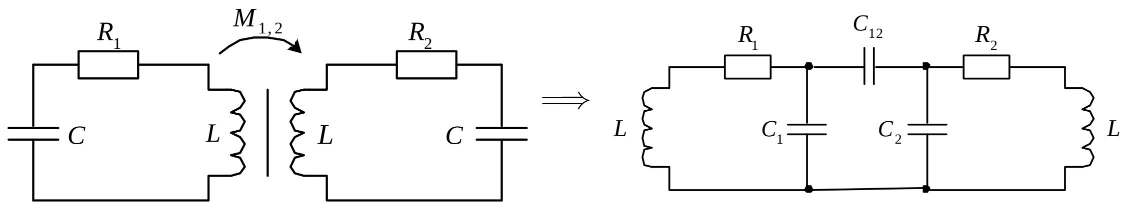
$$M_{1,2} = \Delta \cdot L \cdot x_{1,2}$$

General:

$$\boxed{M_{k,k+1} = \Delta \cdot L \cdot x_{k,k+1}}$$

A bandpass filter with an inductive coupling results.

Application of the capacitive coupling:



$$\omega_m^2 = \frac{1}{L \cdot C}$$

It is valid:

$$C = C_1 + \frac{C_{12} \cdot C_2}{C_{12} + C_2} \text{As } C_{12} << C_2 \quad \Rightarrow \quad \text{Approximation:}$$

$$\boxed{C \approx C_1 + C_{12}}$$

The following equation applies to repeatedly coupled circles (inner circles):

$$\boxed{C \approx C_{k-1,k} + C_k + C_{k,k+1}}$$

Calculation of the coupling capacities:
(Philippow: Taschenbuch der Elektrotechnik, Bd. II, S. 580, Bild 7.145)

It is valid:

$$C_{k,k+1} = C \cdot \frac{\omega_m \cdot M_{k,k+1} \cdot C}{1 - (\omega_m^2 \cdot M_{k,k+1} \cdot C)^2}$$

Conversion:

$$\omega_m^2 \cdot M_{k,k+1} \cdot C = \omega_m^2 \cdot x_{k,k+1} \cdot \Delta \cdot L \cdot C = \Delta \cdot x_{k,k+1}$$

With that:

$$\boxed{C_{k,k+1} = C \cdot \frac{\Delta \cdot x_{k,k+1}}{1 - (\Delta \cdot x_{k,k+1})^2}}$$

Numerical values for the example::
(Denormalizing)

$$\mathrm{C} = \frac{1}{\omega_m^2 \cdot L} = \underline{\underline{6{,}332574\ \mathrm{nF}}} \qquad C_{1,2} = \underline{\underline{90{,}3057\ \mathrm{pF}}}$$

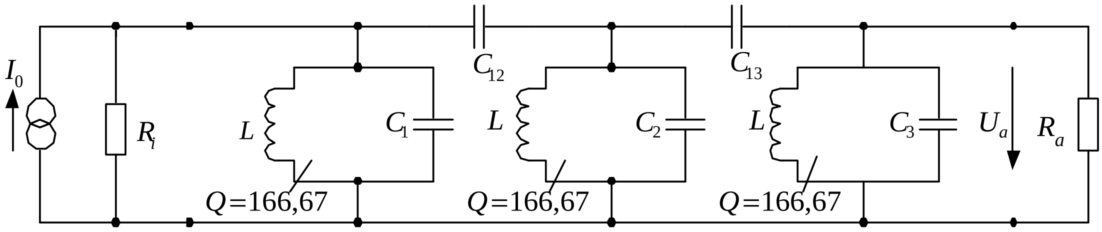
$$C_{2,3} = \underline{\underline{85{,}2841\ \mathrm{pF}}}$$

$$C_1 = C - C_{1,2} = \underline{\underline{6{,}2422682\ \mathrm{nF}}}$$

$$C_2 = C - C_{1,2} - C_{2,3} = \underline{\underline{6{,}1569841\ \mathrm{nF}}}$$

$$C_3 = C - C_{2,3} = \underline{\underline{6{,}2472898\ \mathrm{nF}}}$$

Circuit diagram with source and load:
C-coupling



All circles get the same swinging Q factor. The greater attenuations for the circles 1 and 3 are realized by the input resistor of the source and the load resistor (input resistor of the following amplifier).

Required quality factor of the 1st resonant circuit: $Q_1 = 1/\Delta \cdot \delta_1$
The inner resonant circuits have a quality factor $Q_0 > Q_1$

It is valid: $\left.\begin{array}{l} R_{0p} = Q_0 \cdot \omega_m \cdot L \\ R_{1p} = Q_1 \cdot \omega_m \cdot L \end{array}\right\}$ $\quad R_{1p} = R_{0p}||R_i \quad \Longrightarrow \quad R_i = \dfrac{R_{0p} \cdot R_{1p}}{R_{0p} - R_{1p}}$

$$R_i = \frac{Q_0 \cdot \omega_m \cdot L \cdot Q_1 \cdot \omega_m \cdot L}{\omega_m \cdot L \cdot (Q_0 - Q_1)} = \omega_m \cdot L \cdot \frac{Q_0 \cdot Q_1}{Q_0 - Q_1}$$
$$= \omega_m \cdot L \cdot \frac{\frac{1}{\Delta \cdot \delta_0} \cdot \frac{1}{\Delta \cdot \delta_1}}{\frac{1}{\Delta \cdot \delta_0} - \frac{1}{\Delta \cdot \delta_1}} = \frac{\omega_m \cdot L}{\Delta(\delta_1 - \delta_0)} = \frac{2\Pi \cdot 2 \cdot 10^5 \cdot 10^-4}{4/200 \cdot (0,409579 - 0,3)} = \underline{\underline{57,3393 \text{ k}\Omega}}$$

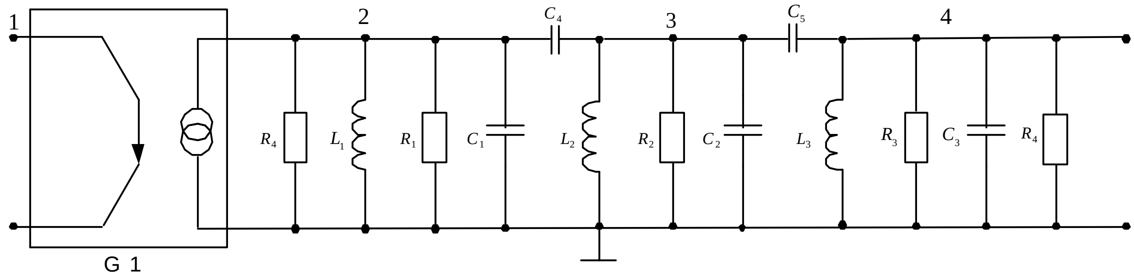
It is valid correspondingly:

$$R_a = \frac{\omega_m \cdot L}{\Delta \cdot (\delta_3 - \delta_0)} = \underline{\underline{6,34395 \text{ k}\Omega}}$$

Testing the ready filter circuit with the help of a network analyzer software, e. g.

- PSpice
- Design-Center (PSpice für Windows)

Processing of the circuit diagram for PSpice:

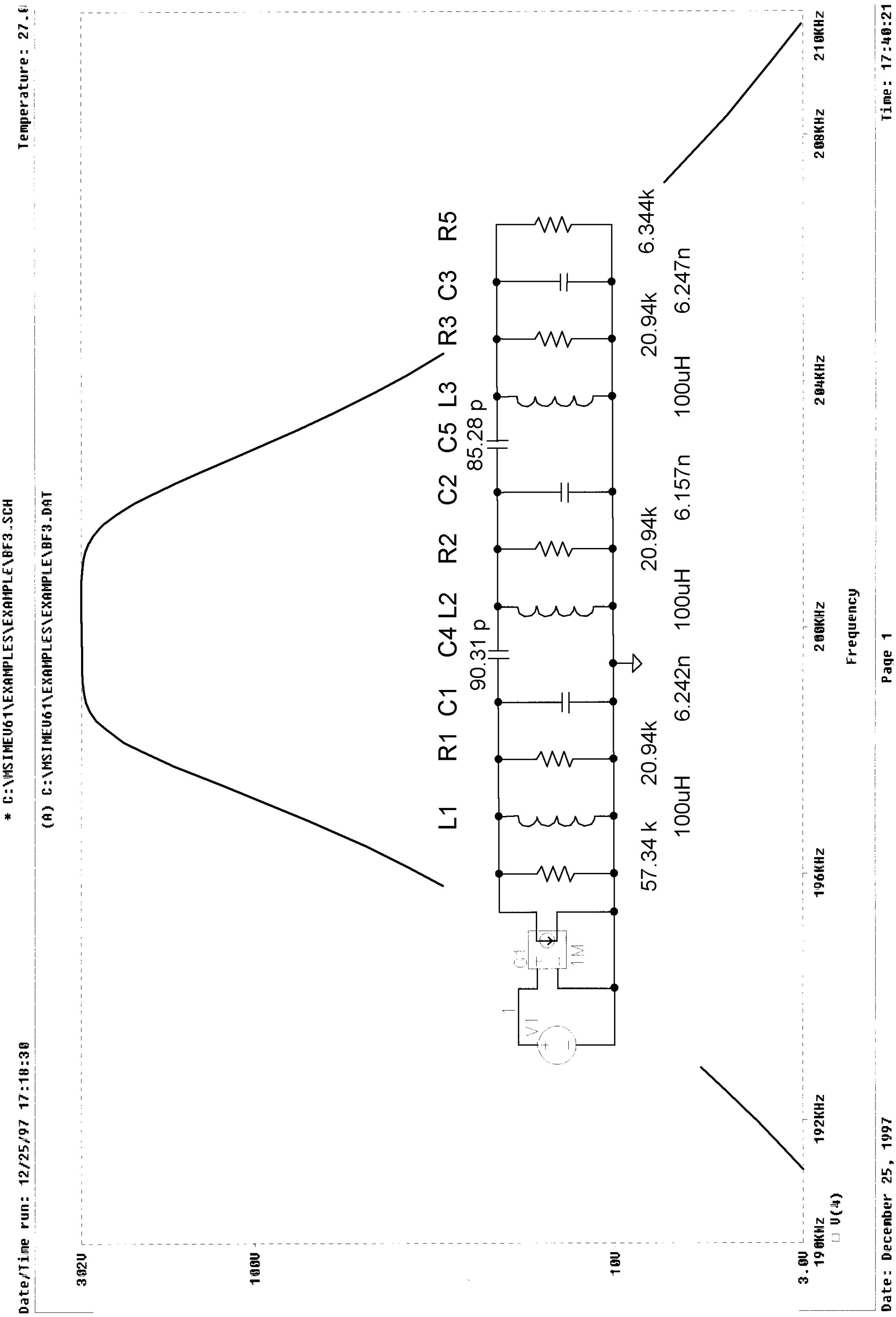


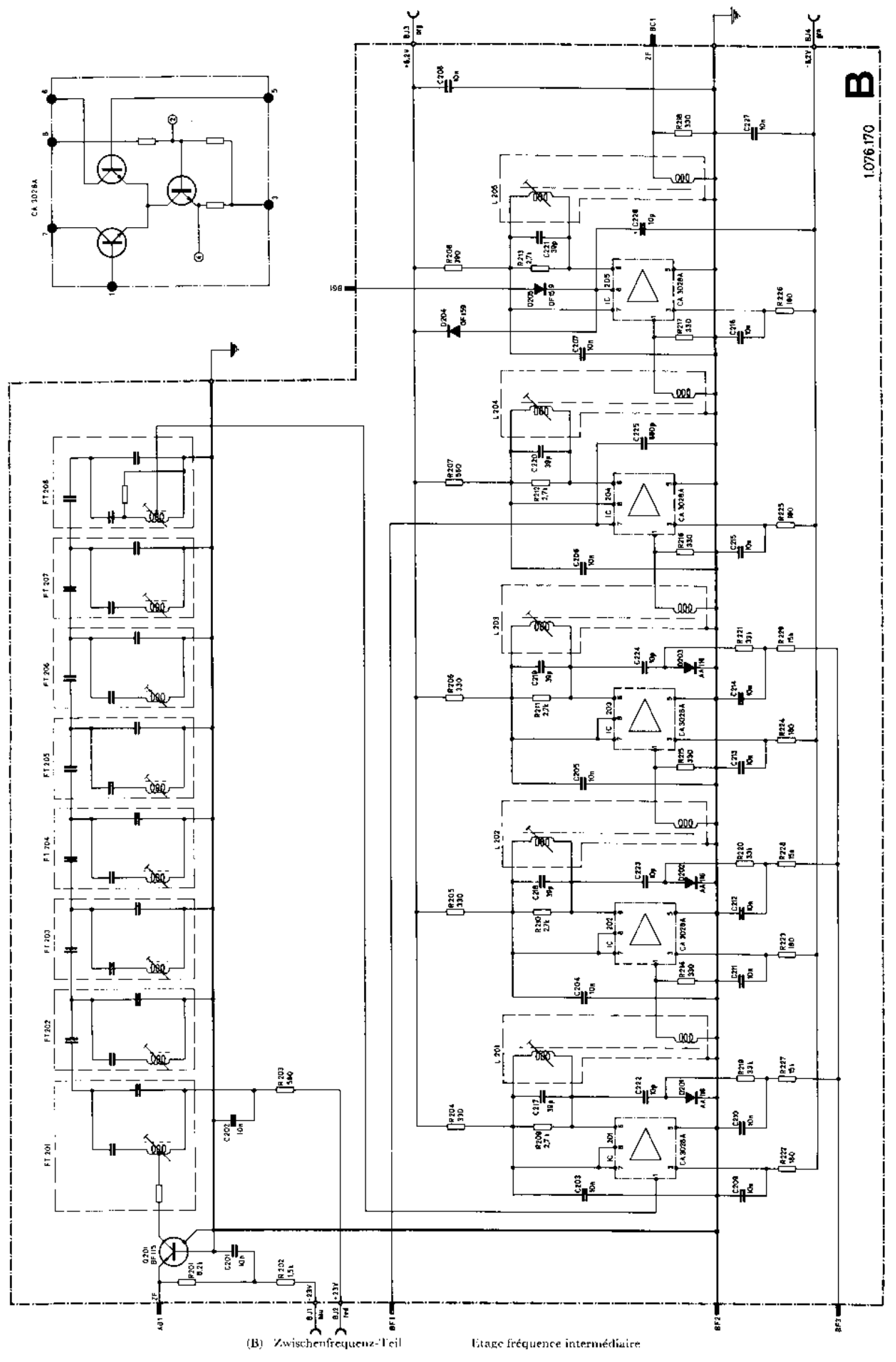
Electrical devices:
Resonant impedances of the oscillating circuits:

$$R_p = \omega_0 \cdot L \cdot Q = 2\Pi \cdot 2 \cdot 10^5 \cdot 10^{-4} \cdot 166,67 = 4\pi \cdot 10 \cdot 166,67$$
$$R_p = 20,944\ \text{k}\Omega$$

List of devices:

L_1	$=$	$L_2 = L_3 = 0{,}1$ mH
R_1	$=$	$R_2 = R_3 = 20{,}94$ kΩ
R_4	$=$	57,34 kΩ
R_5	$=$	6,344 kΩ
C_1	$=$	6,242 nF
C_2	$=$	6,157 nF
C_3	$=$	6,24 nF
C_4	$=$	90,21 pF
C_5	$=$	85,28 pF


Date/Time run: 12/25/97 17:10:30
* C:\MSIMEV61\EXAMPLES\EXAMPLE\BF3.SCH
(A) C:\MSIMEV61\EXAMPLES\EXAMPLE\BF3.DAT
Temperature: 27.0
302V
100V
10V
3.0V
190KHz
192KHz
196KHz
200KHz
204KHz
208KHz
210KHz
□ V(4)
Frequency
Date: December 25, 1997
Page 1
Time: 17:40:21
L1 R1 C1 C4 L2 R2 C2 C5 L3 R3 C3 R5
90.31 p
85.28 p
57.34 k
100uH
20.94k
6.242n
100uH
20.94k
6.157n
100uH
20.94k
6.247n
6.344k
V1
C1
1M



Application of compact filter in the IF-amplifier of the HiFi-Tuner ReVox A76. It is used an eight stage Gauss filter with linear phase characteristic.