On the Approximate Parametrization of Perturbed Algebraic Curves and Surfaces

SAGA — Symbolic Algebraic Group in Alcala

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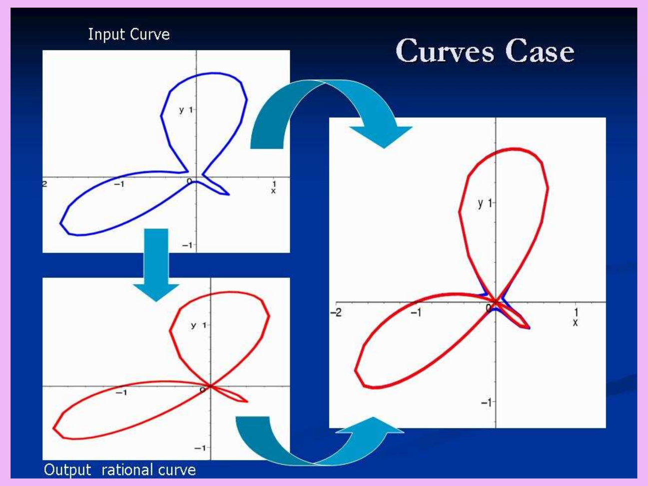
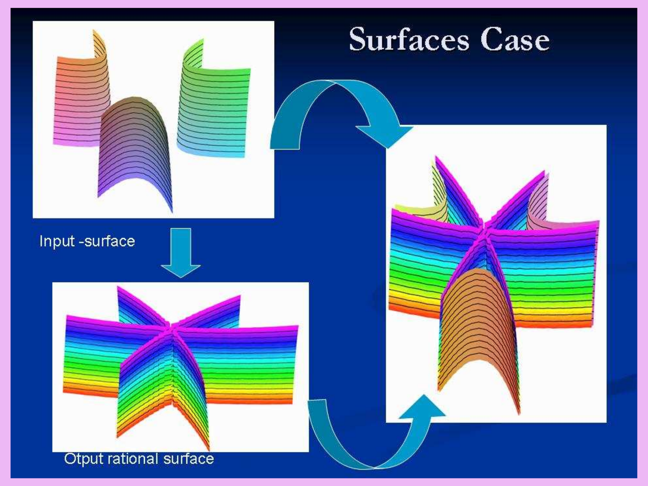
Statement of the Problems

Approximate Global Parametrization

Finite Piecewise Polynomial Parametrization of Rational Curves





Approximate Global Parametrization

- **GIVEN** an input perturbed algebraic variety $\mathcal{V}$
- **COMPUTE** a proper rational parametrization $\overline{\mathcal{P}}(\bar{t})$ of $\overline{\mathcal{V}}$ such that $\overline{\mathcal{V}}$ lies in the "*vicinity*" of $\mathcal{V}$

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VICINITY

Let ϵ be a given tolerance, $\mathcal{V}$ and $\overline{\mathcal{V}}$ are close if and only if, $\overline{\mathcal{V}}$ is in the offset region of $\mathcal{V}$ at distance $2\delta(\epsilon)$.

Reference for this notion: [1]

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Main Notion: ϵ-SINGULARITY

We say that $\overline{P} \in \mathbb{C}^n$ is an ϵ–(affine) point of an algebraic hypersurface $\mathcal{V}$ defined over $\mathbb{C}$ by a polynomial $f(x_1,\dots,x_n) \in \mathbb{C}[x_1,\dots,x_n]$ of degree d, if there exists $r \in \mathbb{N}$, $1 \le r \le d$, such that
(1.) for $0 \le i_1+\dots+i_n \le r-1$, it holds that $\left|f^{i_1,\dots,i_n}(\overline{P})\right| < \epsilon \cdot \|f\|$,
(2.) there exist $i_1^0,\dots,i_n^0 \in \mathbb{N}$ with $i_1^0+\dots+i_n^0 = r$ such that $\left|f^{i_1^0,\dots,i_n^0}(\overline{P})\right| \ge \epsilon \cdot \|f\|$.
If $r=1$ we say that $\overline{P}$ is an ϵ–(affine) simple point. Otherwise, we say that $\overline{P}$ is an ϵ–(affine) singularity of multiplicity r.

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Main Results

- Let $\overline{P} \in \mathbb{C}^n$ be an ϵ–singularity of multiplicity r of $\mathcal{V}$ of degree d. There exist at least r points $P_1,\dots,P_r \in \mathcal{V}$ such that

$$||\overline{P}-P_j||_2 \le 6(r!\cdot \epsilon^{\mathrm{depth}(\overline{P})-\mathrm{height}(\overline{P})})^{1/r}, \qquad j=1,\dots,r.$$

  Parameter *depth* measures how close the ϵ–point of multiplicity r is to be an exact singularity of multiplicity r, while *height* measures how close the ϵ–point of multiplicity r is to be an ϵ–point of multiplicity $r+1$.

  Reference for this result: [4]

- Let $\mathcal{V} \in S_\epsilon^d$, where S_ϵ^d is the set of all the real algebraic varieties of degree d that have an ϵ–singularity of multiplicity $d-1$.
  (1) If $\mathcal{V}$ is a **curve**, then $\mathcal{V} \subset \mathcal{O}_{2\sqrt{2}\epsilon^{\frac{1}{2d}}\exp(2)}(\overline{\mathcal{V}})$ and $\overline{\mathcal{V}} \subset \mathcal{O}_{2\sqrt{2}\epsilon^{\frac{1}{2d}}\exp(2)}(\mathcal{V})$.
  (2) If $\mathcal{V}$ is a **surface**, then $\mathcal{V} \subset \mathcal{O}_{3\sqrt{3}\epsilon^{\frac{1}{2d}}\exp(3)}(\overline{\mathcal{V}})$ and $\overline{\mathcal{V}} \subset \mathcal{O}_{3\sqrt{3}\epsilon^{\frac{1}{2d}}\exp(3)}(\mathcal{V})$.

Reference for these results: [2, 3]

GEOMETRIC IDEA FOR CURVES

Let $\mathcal{C}$ be a curve of degree d, having an affine ϵ-singularity, $\overline{P}=(\overline{a},\overline{b})$, of multiplicity $d-1$. We consider a pencil of lines $\mathcal{H}_t$ passing through the ϵ–singularity $\overline{P}$ of $\mathcal{C}$ of multiplicity $d-1$:

$$\mathcal{H}_t(x,y,t) = y - tx - \overline{b} + \overline{a}t.$$

If $\overline{P}$ would have been an exact singularity, then the SYMBOLIC ALGORITHM would have output the parametrization $\overline{\mathcal{P}}(t) = (\overline{p}_1(t), \overline{p}_2(t))$, where $\overline{p}_1(t)$ is the root of the univariate polynomial

$$\frac{f(x, tx+\overline{b}-\overline{a}t)}{(x-\overline{a})^{d-1}}$$

and $\overline{p}_2(t) = t\overline{p}_1(t) + \overline{b} - t\overline{a}$.

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In our case $\overline{P}$ is not a singularity, but an ϵ–singularity.
Then, the GEOMETRIC IDEA consists in computing the root of the quotient of $f(x, tx+\overline{b}-\overline{a}t)$ and $(x-\overline{a})^{d-1}$ w.r.t x. Let $\overline{p}_1(t)$ be this root. Then, we prove that

$$\overline{\mathcal{P}}(t) = (\overline{p}_1(t), t\overline{p}_1(t) + \overline{b} - t\overline{a})$$

is an approximate parametrization of $\mathcal{C}$. And the implicit equation of $\overline{\mathcal{C}}$ defined by the parametrization $\overline{\mathcal{P}}(t)$ is

$$\overline{f}(x,y) = f(x,y) - T(x,y)$$

where $f(x,y)$ the implicit equation of $\mathcal{C}$ and $T(x,y)$ is the Taylor expansion up to order $d-1$ of $f(x,y)$ at $\overline{P}$.

APPROXIMATE PARAMETRIZATION OF CURVES

- **GIVEN** a tolerance $\epsilon > 0$ and a polynomial $f(x,y) \in \mathbb{C}[x,y]$, defining implicitly a curve $\mathcal{C}$ having an affine ϵ-singularity of maximum multiplicity
- **COMPUTE** a rational parametrization $\overline{\mathcal{P}}(t)$ of a rational curve $\overline{\mathcal{C}}$ that is contained in the offset region of $\mathcal{C}$ (and reciprocally) at distance $2\sqrt{2}\cdot\epsilon^{\frac{1}{2d}}\cdot e^2$.
  (1) Compute an affine ϵ–singularity, $\overline{P}=(\overline{a},\overline{b})$, of $\mathcal{C}$ of multiplicity $d-1$.
  (2) Compute $\overline{f}(x,y) = f(x,y) - T(x,y)$, where $T(x,y)$ is the Taylor expansion of $f(x,y)$ up to order $d-1$ at $\overline{P}$.
  (3) Compute

$$A(x,y,t) = \frac{\displaystyle\sum_{r=0}^{d-1} \frac{\frac{\partial^{d-1}\overline{f}}{\partial^r x \partial^{d-1-r} y} t^r}{(d-1-r)!r!}}{\displaystyle\sum_{r=0}^{d} \frac{\frac{\partial^{d}\overline{f}}{\partial^r x \partial^{d-r-s} y} t^r}{(d-r)!r!}}$$

  and return $\overline{\mathcal{P}}(t) = \left(-A(\overline{P},t)+\overline{a}, -tA(\overline{P},t)+\overline{b}\right)$.

Reference for this result: [2]

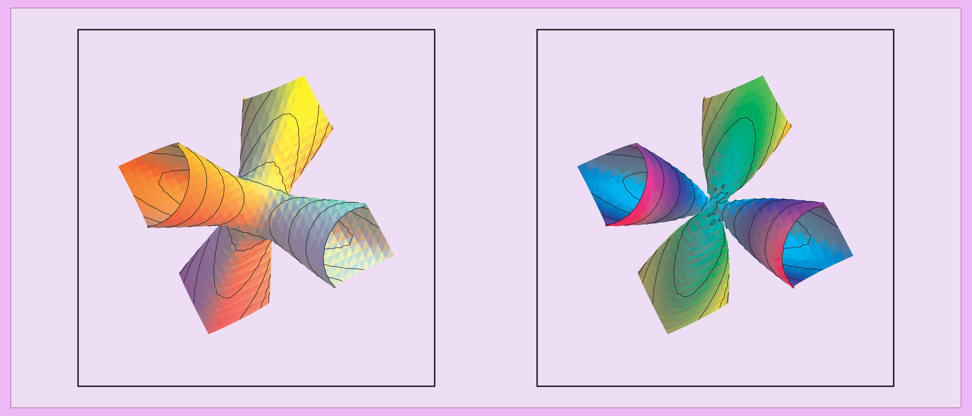
APPROXIMATE PARAMETRIZATION FOR SURFACES

- **GIVEN** a tolerance $\epsilon > 0$ and a polynomial $f(x,y,z) \in \mathbb{C}[x,y,z]$, defining implicitly a surface V with an ϵ–singularity of maximum multiplicity,
- **COMPUTES** a parametrization $\overline{\mathcal{P}}$ of a rational surface $\overline{V}$ that is contained in the offset region of V (and reciprocally) at distance $3\sqrt{3}\cdot\epsilon^{\frac{1}{2d}}\cdot e^3$.
  (1) Compute an affine ϵ–singularity, $\overline{P}=(\overline{a},\overline{b},\overline{c})$, of V of multiplicity $d-1$.
  (2) Compute $\overline{f}(x,y,z) = f(x,y,z) - T(x,y,z)$, where $T(x,y,z)$ is the Taylor expansion of $f(x,y,z)$ up to order $d-1$ at $\overline{P}$.
  (3) Compute

$$A(x,y,z,t,h) = \frac{\displaystyle\sum_{r+s=0}^{d-1} \frac{\frac{\partial^{d-1}\overline{f}}{\partial^r x \partial^s y \partial^{d-1-r-s} z} t^r h^s}{(d-1-r-s)!r!s!}}{\displaystyle\sum_{r+s=0}^{d} \frac{\frac{\partial^{d}\overline{f}}{\partial^r x \partial^s y \partial^{d-r-s} z} t^r h^s}{(d-r-s)!r!s!}}$$

  and return $\overline{\mathcal{P}}(t,h) = \left(-tA(\overline{P},t,h)+\overline{a}, -hA(\overline{P},t,h)+\overline{b}, -A(\overline{P},t,h)+\overline{c}\right)$.

Reference for this result: [3]



Finite Piecewise Polynomial Parametrization of Rational Curves

- **GIVEN** a non-polynomial rational parametrization $\mathcal{P}(t) \in \mathbb{R}(t)^2$, and $\epsilon > 0$.
- **DETERMINE** a partition $\mathcal{D}$ of the parameter space as a union of finitely many intervals, and for each $I \in \mathcal{D}$, the algorithm generates a polynomial parametrization $\mathcal{P}_I(t)$ such that $\mathcal{C}_I = \{\mathcal{P}(t)\,|\,t \in I\}$ is in the offset region of $\mathcal{C}_I^\star = \{\mathcal{P}_I(t)\,|\,t\in I\}$, at distance at most $\sqrt{2}\epsilon$, and conversely. Moreover, one may also input a natural number N and then the algorithm returns polynomial parametrizations with degrees smaller or equal to N.

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Main Notions

Let $\chi(t) = \frac{\chi_1(t)}{\chi_2(t)} \in \mathbb{R}(t)\setminus\mathbb{R}[t]$ be continuous in a compact interval $K \subset \mathbb{R}$ in reduced form, and $\chi_2(t) > 0$ for all $t \in K$. Let $\chi(t) = c(t) + \frac{r(t)}{\chi_2(t)}$, where $c(t)$ and $r(t)$ are the quotient and the remainder of $\chi_1(t)$ divided by $\chi_2(t)$. Let $x_0^\star = \frac{1}{2}\left(M_{\frac{r(t)}{\chi_2(t)},K} + m_{\frac{r(t)}{\chi_2(t)},K}\right)$ where $M_{f(t),K}$ and $m_{f(t),K}$ denote the maximum and the minimum of $f(t)$ in K, resp.

(1.) The approximating polynomial sequence $\{P^{\chi(t),K}(t,n)\}_{n\in\mathbb{N}_0}$ associated to $\chi(t)$ in K is

$$P^{\chi(t),K}(t,n) = c(t) + x_0^\star + \frac{r(t)-x_0^\star\chi_2(t)}{M_{\chi_2,K}}\sum_{k=0}^{n}\left(1-\frac{\chi_2(t)}{M_{\chi_2,K}}\right)^k.$$

(2.) The approximating error sequence $\{\mathcal{R}^{\chi(t),K}(t,n)\}_{n\in\mathbb{N}_0}$ associated to $\chi(t)$ in K is $R^{\chi(t),K}(t,n) = \chi(t) - P^{\chi(t),K}(t,n)$.

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Main Result

(1.) The k-th derivative sequence of $\{P^{\chi(t),K}(t,n)\}_{n\in\mathbb{N}_0}$ converges uniformly to $\chi^{k)}(t)$ in K for all $k \in \mathbb{N}_0$.

(2.) $\forall t \in K$, $|\mathcal{R}^{\chi(t),K}(t,n)| \le \frac{1}{2}\left(M_{\frac{r(t)}{\chi_2(t)},K} - m_{\frac{r(t)}{\chi_2(t)},K}\right)\left(1-\frac{m_{\chi_2(t),K}}{M_{\chi_2(t),K}}\right)^{n+1}$

In order to find in $\{P^{\chi(t),K}(t,n)\}_{n\in\mathbb{N}}$ the polynomial approximating to $\chi(t)$ in K with less degree, we define:

THE ϵ-ORDER OF CONVERGENCE OF $\{P^{\chi(t),K}(t,n)\}_{n\in\mathbb{N}_0}$ IN K, denoted by $\theta_\epsilon(P^{\chi(t),K}(t,n))$, is the smallest $n_0 \in \mathbb{N}_0$ such that $|\chi(t) - P^{\chi(t),K}(t,n)| \le \epsilon$ for all $n \ge n_0$ and for all $t \in K$. We estimate θ_ϵ computing $n^\star \in \mathbb{N}_0$ such that $\frac{1}{2}\left(M_{\frac{r(t)}{\chi_2(t)},K} - m_{\frac{r(t)}{\chi_2(t)},K}\right)\left(1-\frac{m_{\chi_2(t),K}}{M_{\chi_2(t),K}}\right)^{n^\star+1} \le \epsilon$. We denote the output by $\mathsf{SAlg}_1(\epsilon,\chi,K)$; see Step 3 in Algorithm 1 .

Based on these results we derive an algorithm for approximating polynomially a rational function in a compact:

ALGORITHM 1: POLYNOMIAL APPROXIMATION IN A COMPACT. We denote the output by $\mathsf{Alg}_1(\epsilon,\chi,K)$.

INPUT $(\epsilon,\chi(t),K)$ where $\epsilon>0$, $\chi(t) = \frac{\chi_1(t)}{\chi_2(t)} \in \mathbb{R}(t)\setminus\mathbb{R}[t]$, and $K \subset \mathbb{R}$ is a compact where $\chi(t)$ is continuous.

OUTPUT a polynomial approximating $\chi(t)$ in K under precision ϵ.

1. Take $t_0 \in K$. IF $\chi_2(t_0) < 0$ THEN for $i=1,2$ replace χ_i by $-\chi_i$. IF $\deg(\chi_1) < \deg(\chi_2)$ THEN $c := 0, r := \chi_1$ ELSE take c, r as the quotient and the remainder of χ_1 divided by χ_2.
2. $M := M_{\frac{r(t)}{\chi_2(t)},K}$, $m := m_{\frac{r(t)}{\chi_2(t)},K}$, $M^* := M_{\chi_2(t),K}$, $m^* := m_{\chi_2(t),K}$, $x_0 := \frac{M+m}{2}$, $\alpha := \frac{M-m}{2}$, $\beta := \frac{M^*-m^*}{M^*}$.
3. [Estimation of the ϵ-order of convergence $\theta_\epsilon(P^{\chi(t),K}(t,n))$] IF K is a point, THEN RETURN $\theta_\epsilon = 0$. ELSE RETURN $\theta_\epsilon = \max\{0, \lceil\log_\beta(\epsilon/\alpha)-1\rceil\}$.
3. RETURN $\mathbf{P}^{\chi(t),K}(\mathbf{t},\theta_\epsilon) := \mathbf{c}(\mathbf{t}) + \mathbf{x_0} + \frac{\mathbf{r}(\mathbf{t}) - \mathbf{x_0}\,\chi_2(\mathbf{t})}{\mathbf{M}^*}\sum_{\mathbf{k}=0}^{\theta_\epsilon}\left(1-\frac{\chi_2(\mathbf{t})}{\mathbf{M}^*}\right)^{\mathbf{k}}$.

Reference for these results: [5]

Degree Control of the Approximation.

We can decompose the given compact K as union of finitely many compact intervals K_i such that the degree of the approximating polynomial is smaller or equal to a given natural number N, which satisfies that:

- $N \in \mathbb{N}$ is such that $N \ge \max\{\deg(\chi_1)-\deg(\chi_2), \deg(\chi_2)\}$.
- Associated with N we consider the auxiliary number $N_1 := \left\lfloor -1 + \frac{N}{\deg(\chi_2)}\right\rfloor$

The relation between N and N_1 is that if $p(t) := \mathsf{Alg}_1(\epsilon,\chi(t),K)$ and $\theta_\epsilon(P^{\chi(t),K}(t,n)) \le N_1$ then $\deg(p(t)) \le N$.

Therefore, assuming that $\theta_\epsilon(P^{\chi(t),K}(t,n)) > N_1$ the strategy consists in decomposing K as $K := [\lambda_1,\lambda_2]\cup\dots\cup[\lambda_\ell,\lambda_{\ell+1}]$ and such that $\theta_\epsilon(P^{\chi(t),[\lambda_i,\lambda_{i+1}]}(t,n)) \le N_1$. The next lemma shows how to proceed.

LEMMA. Let $K' = [\gamma_1,\gamma_2] \subset K$. If $\theta_\epsilon(P^{\chi(t),K'}(t,n)) > N_1$, there exists a unique $\gamma \in (\gamma_1,\gamma_2)$ such that $\theta_\epsilon(P^{\chi(t),[\gamma_1,\gamma]}(t,n)) = N_1$. Furthermore, if $x \in (\gamma_1,\gamma)$ then $\theta_\epsilon(P^{\chi(t),[\gamma_1,x]}(t,n)) \le N_1$. γ is the real zero of the function $\mathfrak{T}_{K'}(x) := \alpha(x)\beta(x)^{N_1+1} - \epsilon$ where $K'(x) := [\gamma_1,x]$, $\alpha(x) := \frac{1}{2}\left(M_{\frac{r}{\chi_2},K'(x)} - m_{\frac{r}{\chi_2},K'(x)}\right)$ y $\beta(x) := 1 - \frac{m_{\chi_2,K'(x)}}{M_{\chi_2,K'(x)}}$.

The following algorithm computes γ when $r(t)/\chi_2(t)$ and $\chi_2(t)$ are monotone in K'.

COMPUTATION OF γ. We denote the output by $\mathsf{SAlg}_2(\epsilon,\chi,N,K')$.

INPUT $(\epsilon,\chi(t),N,K')$ where $\chi(t) = c(t) + \frac{r(t)}{\chi_2(t)}$, $N \in \mathbb{N}$ is such that $N \ge \max\{\deg(\chi_1(t))-\deg(\chi_2(t)), \deg(\chi_2(t))\}$; $K' = [\gamma_1,\gamma_2] \subset \mathbb{R}$ is such that $\chi_2(t) > 0$, $\frac{r}{\chi_2}$ and χ_2 are monotone in K'; $\theta_\epsilon(P^{\chi(t),K'}(t,n)) > N_1$, where $N_1 := \left\lfloor -1 + \frac{N}{\deg(\chi_2)}\right\rfloor$.

OUTPUT $\gamma \in (\gamma_1,\gamma_2)$ such that $\theta_\epsilon(P^{\chi(t),[\gamma_1,\gamma]}(t,n)) \le N_1$.

1. $\mathcal{T}(x_1,x_2,x_3,x_4) := \frac{1}{2}\left(\frac{r}{\chi_2}(x_1) - \frac{r}{\chi_2}(x_2)\right)\left(1-\frac{\chi_2(x_3)}{\chi_2(x_4)}\right)^{N_1+1} - \epsilon$.
2. IF $\frac{r}{\chi_2}$, χ_2 are increasing in K' THEN approximate the root γ in K' of the numerator of $\mathcal{T}(x,\gamma_1,\gamma_1,x)$. RETURN γ
3. IF $\frac{r}{\chi_2}$, χ_2 are decreasing in K' THEN approximate the root γ in K' of the numerator of $\mathcal{T}(\gamma_1,x,x,\gamma_1)$. RETURN γ
4. IF $\frac{r}{\chi_2}$ is increasing, and χ_2 is decreasing in K' THEN approximate the root γ in K' of the numerator of $\mathcal{T}(x,\gamma_1,x,\gamma_1)$
5. IF $\frac{r}{\chi_2}$ is decreasing, and χ_2 is increasing in K' THEN approximate the root γ in K' of the numerator of $\mathcal{T}(\gamma_1,x,\gamma_1,x)$

On-Going Research. Currently, we are studying the extension of the methods described above. Concerning to the first problem we have derived an approximate algorithm for parametrizing perturbed rational curves with arbitrary ϵ-singularity locus. Nevertheless, although experimental analysis seems to show that the output is quite satisfactory, we still do not have a complete theoretical error analysis of the algorithm. Concerning to second problem, we are dealing with the problem of extending the ideas to the plane implicit case and the space (parametric/implicit) case.

Partition of the Parameter Space and Polynomial Assignments.

Partition of the Parameter Space: The Non-Bounded Part.

For $\chi(t) = c(t) + r(t)/\chi_2(t)$, as $\lim_{t\to\infty}\frac{r(t)}{\chi_2(t)} = 0$, there exists $B_{\chi,\epsilon} \in \mathbb{R}^+$ such that for every $t \in \mathbb{R}$ with $|t| > B_{\chi,\epsilon}$, $\left|\frac{r(t)}{\chi_2(t)}\right| \le \epsilon$. Thus, the unbounded part of the decomposition is taken as $I_0 = (-\infty,-B_{\chi,\epsilon})\cup(B_{\chi,\epsilon},\infty)$

COMPUTATION OF $B_{\chi,\epsilon}$. We denote the output by $\mathsf{SAlg}_3(\epsilon,\chi(t))$.

INPUT $(\epsilon,\chi(t))$ where $\epsilon>0$, $\chi(t) = \frac{\chi_1(t)}{\chi_2(t)} = c(t) + \frac{r(t)}{\chi_2(t)} \in \mathbb{R}(t)$ with $c, r, \chi_2 \in \mathbb{R}[t]$, $\gcd(r,\chi_2) = 1$, $\deg(r) < \deg(\chi_2)$.

OUTPUT $B \in \mathbb{R}^+$ such that for every $t \in \mathbb{R}$ with $|t| > B$ it holds that $\left|\frac{r(t)}{\chi_2(t)}\right| \le \epsilon$.

- Compute an upper bound B^+ of the roots, in module, of the polynomial $r(t)+\epsilon\chi_2(t)$ as well as an upper bound B^- of the roots, in module, of the polynomial $r(t)-\epsilon\chi_2(t)$. RETURN $B := \max\{B^+,B^-\}$.

Partition of the Parameter Space : The Bounded Part: $\mathbb{R}\setminus I_0 = [-B_{\chi,\epsilon}, B_{\chi,\epsilon}]$.

- IF $\chi_2(t)$ has not real roots: $\mathbb{R} = I_0 \cup K_1$, where $K_1 = [-B_{\chi,\epsilon}, B_{\chi,\epsilon}]$ and $B_{\chi,\epsilon} := \mathsf{SAlg}_3(\epsilon,\chi)$.
- IF $\chi_2(t)$ has real roots ξ_i, $i=1,\dots,m$, we isolate them

$$-B_{\chi,\epsilon} < \alpha_1 < \xi_1 < \beta_1 < \dots < \alpha_m < \xi_m < \beta_m < B_{\chi,\epsilon} \text{ and } \beta_i - \alpha_i \le \epsilon.$$

$[-B_{\chi,\epsilon}, B_{\chi,\epsilon}]$ and $\mathbb{R}$ are decomposed as: $[-B_{\chi,\epsilon}, B_{\chi,\epsilon}] = \bigcup_{i=1}^{m+1} K_i \bigcup_{i=1}^{m} J_i$ and $\mathbb{R} = I_0 \cup [-B_{\chi,\epsilon}, B_{\chi,\epsilon}]$ where $K_i = [\beta_{i-1},\alpha_i]$ are compacts and the rational function is defined, and each $J_i = (\alpha_i,\beta_i)$ isolates one real root of $\chi_2(t)$, namely ξ_i. Here we consider that this length is taken smaller or equal to the tolerance ϵ. The assignments are

$I_0 \longrightarrow c(t)$, $K_i \longrightarrow \mathsf{Alg}_1(\epsilon,\chi,K_i)$, $J_i \longrightarrow$ No assignment.

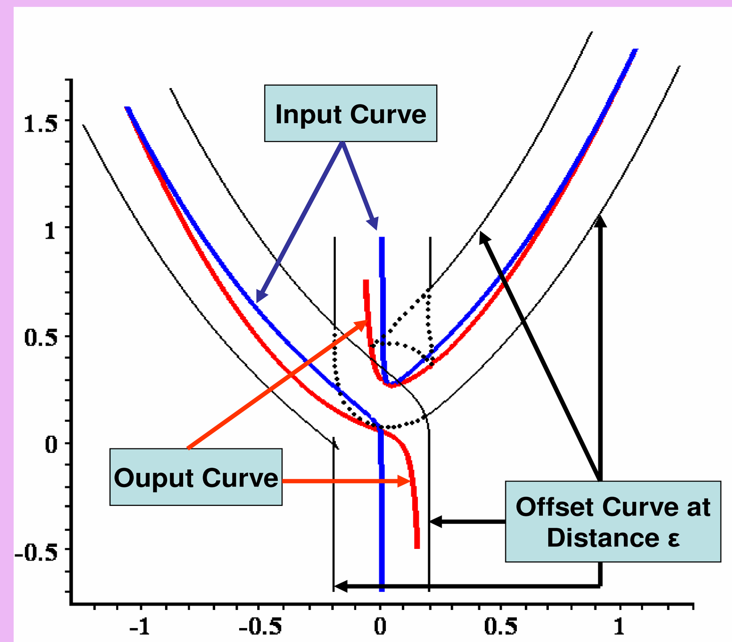
ALGORITHM 2: FINITE PIECEWISE POLYNOMIAL PARAMETRIZATION OF RATIONAL CURVES.

INPUT $(\epsilon,\mathcal{P}(t),N)$ where $\epsilon > 0$, and $\mathcal{P}(t) = (\chi(t),\xi(t)) = \left(\frac{\chi_1(t)}{\chi_2(t)},\frac{\xi_1(t)}{\xi_2(t)}\right)$ is a real rational parametrization of a curve $\mathcal{C}$ non-polynomial,and $N \in \mathbb{N}\cup\{\infty\}$ is such that $N \ge \max\{\deg(\chi_1)-\deg(\chi_2), \deg(\xi_1)-\deg(\xi_2), \deg(\chi_2), \deg(\xi_2)\}$.

OUTPUT a list $\mathcal{F}$. Each element of $\mathcal{F}$ is of the form $[A, \mathcal{Q}(t)]$ where $\mathcal{Q}(t)$ is a polynomial parametrization, and A is a subset of $\mathbb{R}$ where t takes values. Moreover, if $N \in \mathbb{N}$ then $\deg(\mathcal{Q}(t)) \le N$ else, i.e if $N=\infty$, there is no degree control.

1. IF $\deg(\chi_1) < \deg(\chi_2)$ THEN $c_1 := 0$ and $r_1 := \chi_1$, ELSE take c_1, r_1 as the quotient and remainder of χ_1 divided by χ_2. Similarly with ξ generating c_2, r_2.
2. [Non-Bounded Part.] Compute $B := \max\{\mathsf{SAlg}_3(\epsilon,\chi), \mathsf{SAlg}_3(\epsilon,\xi)\}$. Let $I_0 := (-\infty,-B)\cup(B,\infty)$. Append $[I_0,(c_1,c_2)]$ to $\mathcal{F}$.
3. Decide whether $h(t) := \mathrm{lcm}(\chi_2(t),\xi_2(t))$ has real roots or not. $\mathcal{K} := \emptyset$.
4. [Absence of Real Roots: Partition Step.] If $h(t)$ has not real roots DO IF $N=\infty$ THEN append $[-B,B]$ to $\mathcal{K}$ ELSE decompose $[-B,B]$ compact intervals such that $\frac{r_1}{\chi_2}, \frac{r_2}{\xi_2}, \chi_2, \xi_2$ are monotone and append them to $\mathcal{K}$.
5. [Existence of Real Roots: Partition Step.] If $h(t)$ has real roots, isolate them. Let $[-B,B] = \cup_{i=1}^{m+1}K_i \cup_{i=1}^{m} J_i$, where K_i are compacts and J_i the isolating open intervals.IF $N=\infty$ THEN append $K_1,\dots,K_{m+1}$ to $\mathcal{K}$ ELSE decompose each K_i in compact intervals such that $\frac{r_1}{\chi_2}, \frac{r_2}{\xi_2}, \chi_2, \xi_2$ are monotone and append to $\mathcal{K}$.
6. [Approximation with Degree Control.] IF $N=\infty$ THEN $\mathcal{K}' := \mathcal{K}$ ELSE
   6.1. $\mathcal{K}' := \emptyset$, $N_1 := \lfloor -1 + N/\deg(\chi_2)\rfloor$, $N_1^* := \lfloor -1 + N/\deg(\xi_2)\rfloor$.
   6.2. For every $K := [\lambda_1,\mu] \in \mathcal{K}$ DO
      a. $n := \mathsf{SAlg}_1(\epsilon,\chi,[\lambda_1,\mu])$, $n^* := \mathsf{SAlg}_1(\epsilon,\xi,[\lambda_1,\mu])$.
      b. IF $n \le N_1, n^* > N_1^*$ DO $\lambda_2 := \mathsf{SAlg}_2(\epsilon,\xi,N_1^*,K)$, append $[\lambda_1,\lambda_2]$ to $\mathcal{K}'$, replace K by $[\lambda_2,\mu]$ and go to Step a.
      c. IF $n > N_1, n^* \le N_1^*$ DO $\lambda_2 := \mathsf{SAlg}_2(\epsilon,\chi,N_1,K)$, append $[\lambda_1,\lambda_2]$ to $\mathcal{K}'$, replace K by $[\lambda_2,\mu]$ and go to Step a.
      d. IF $n > N_1, n^* > N_1^*$ DO $\lambda_2 := \min\{\mathsf{SAlg}_2(\epsilon,\chi,N_1,K), \mathsf{SAlg}_2(\epsilon,\xi,N_1^*,K)\}$, append $[\lambda_1,\lambda_2]$ to $\mathcal{K}'$, replace K by $[\lambda_2,\mu]$ and go to Step a.
7. [Approximation.] For every $K \in \mathcal{K}'$ DO $p_1(t) := \mathsf{Alg}_1(\epsilon,\chi,K)$, $p_2(t) := \mathsf{Alg}_1(\epsilon,\xi,K)$, append $[K,(p_1,p_2)]$ to $\mathcal{F}$.
8. RETURN $\mathcal{F}$.

Reference for these results: [5]



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