Algorithms for Offset Curves and Surfaces

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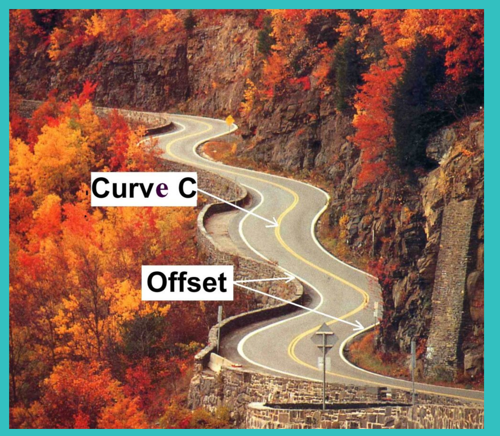
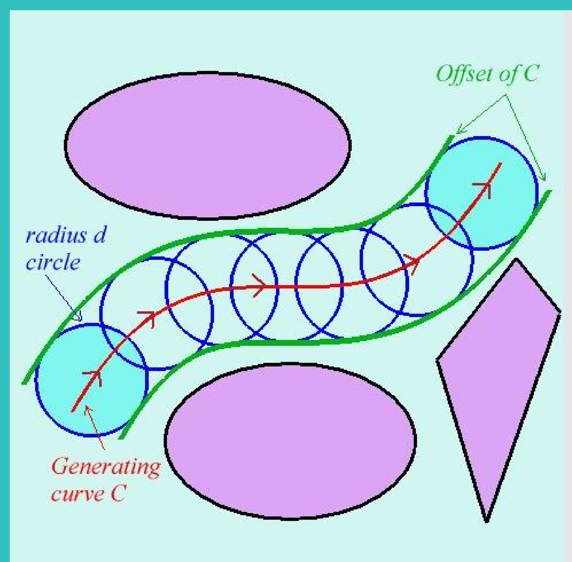


What is an offset?



Suppose that a robot moves around in a room filled with several obstacles. If the robot moves along the curve C, and it is contained in a circle of radius d, then the region swept by the robot is described by the d-offset curve of C, as shown in the figure on the left. Thus, to avoid collisions when planning a trajectory one may compute the offset curve. The offset is closely related to the envolvent of a system of d-circles centered at the points of the curve C.

Similar constructions appear in many applications, such as NC-machining, feature recognition (medial axes transform), and many others (see [4] for a survey).

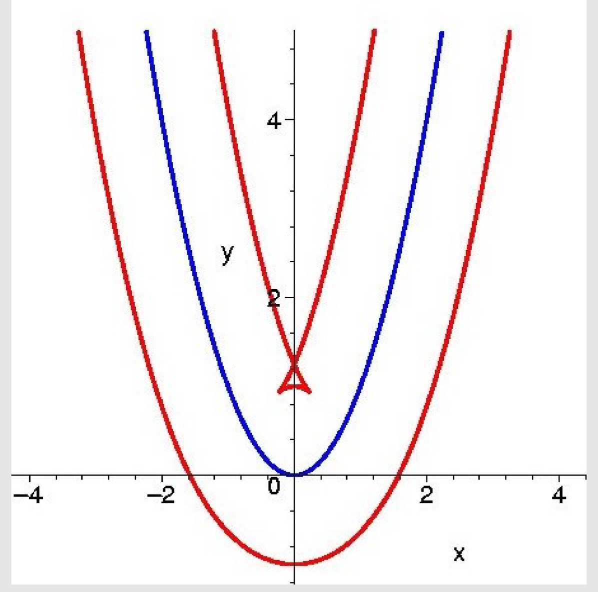
This offsetting proccess can also be applied to surfaces. Offset curves or surfaces are, generally speaking, far more complicated than their generating objects. Pictures at the bottom of the poster show a portion of the offset surface generated by some quadrics.

Classical and Generalized Offsets

Let $\mathcal{P}(\bar t)$ be a parametrization of a hypersurface $\mathcal{V}$, and let $\mathcal{N}(\bar t)$ be the associated normal vector.

Classical d-Offset Parametrization

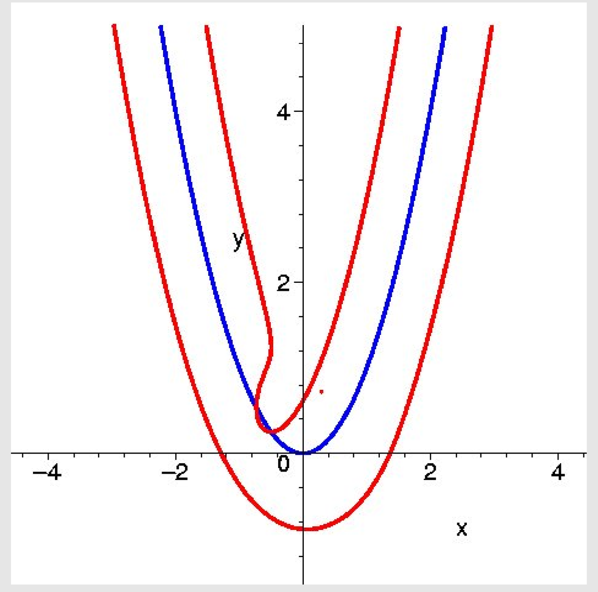
$$\mathcal{P}(\bar t) \pm \frac{d}{\|\mathcal{N}(\bar t)\|}\mathcal{N}(\bar t)$$



Generalized d-Offset Parametrization

Let $A \in O^+(n)$ be a direct isometry.

$$\mathcal{P}(\bar t) \pm \frac{d}{\|\mathcal{N}(\bar t)\|}\mathcal{N}(\bar t) \cdot A$$



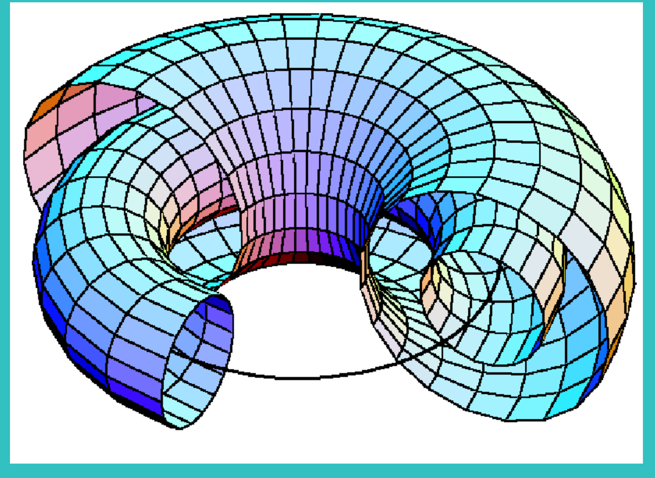
For a formal definition of the (generalized) offset, covering the implicit case, see ([9]). There are other notions of offset, such as the general offsets (see [6]), geodesic offsets, etc.

Offset Atlas

The following table illustrates some properties of the offset, compared with the generating object.

Generating object			Offset	
Name	Degree	Number of terms	Degree	Number of terms
Curves				
Circle	2	3	4	4
Ellipse	2	3	8	15
Hyperbola	2	3	8	12
Parabola	2	2	6	13
Conchoid	3	3	8	24
Cuspidal Cubic	3	2	8	21
Trisectrix	3	4	10	35
Folium	3	3	14	114
Epitrochoid	4	7	10	36
Ramphoid	4	5	14	105
Lemniscate	4	5	12	28
Cardioid	4	6	8	25
Trifolium	4	5	14	63
$x^4+y^4-y^2$	4	3	24	91
Quadrics				
Sphere	2	3	4	7
Cone	2	3	8	29
Circular Cylinder	2	3	4	4
Elliptic Cylinder	2	3	8	15
Circular Ellipsoid	2	4	8	35
Generic Ellipsoid	2	4	12	84
1-sht. Elliptic Hyperboloid	2	4	12	84
1-sht. Circular Hyperboloid	2	4	8	29
2-sht. Elliptic Hyperboloid	2	4	12	84
2-sht. Circular Hyperboloid	2	4	8	29
Circular Paraboloid	2	3	6	26
Elliptic Paraboloid	2	3	10	69
Hyperbolic Paraboloid	2	3	10	71

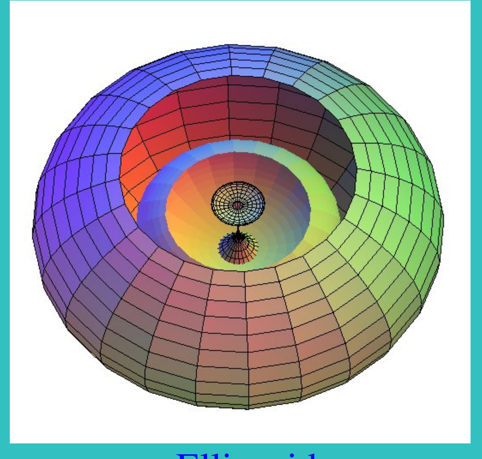
Our web page www2.uah.es/fsegundo/OffsetAtlas (under construction) will soon contain implicit and parametric equations, table of properties, and pictures, for many examples.



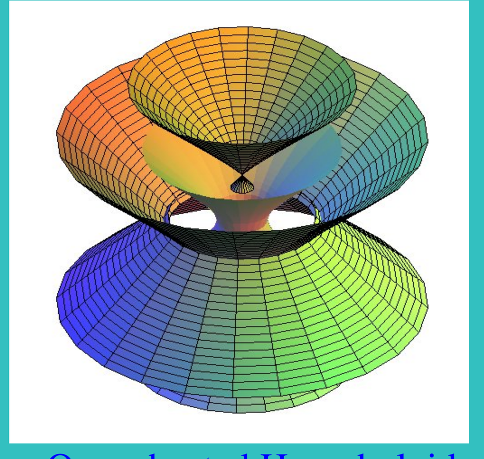
Offset Degeneration

For some values of the distance, as illustrated in the figure on the left for the case of the torus, the offset degenerates: one component of the offset is not a hypersurface. In the classical setting this occurs only for a finite number of distances, that can be algorithmically determined from the generating object. For this result, the extension to generalized offsets, and the associated algorithm see [9].

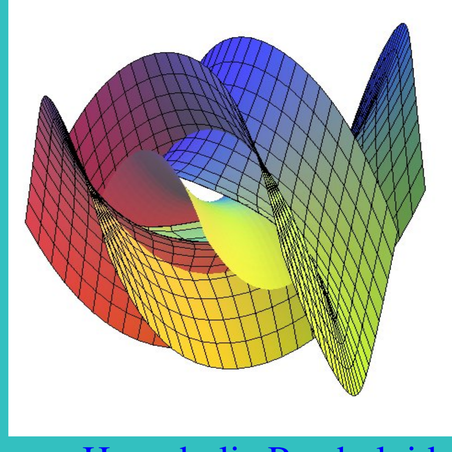
Rationality of the offset for some Quadrics



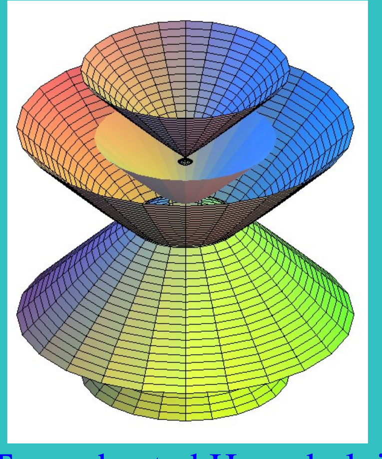
Ellipsoid



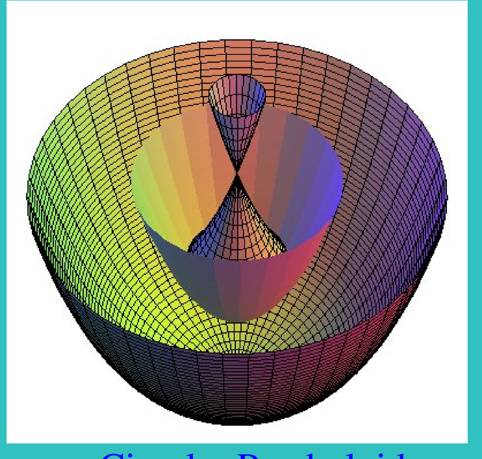
One−sheeted Hyperboloid



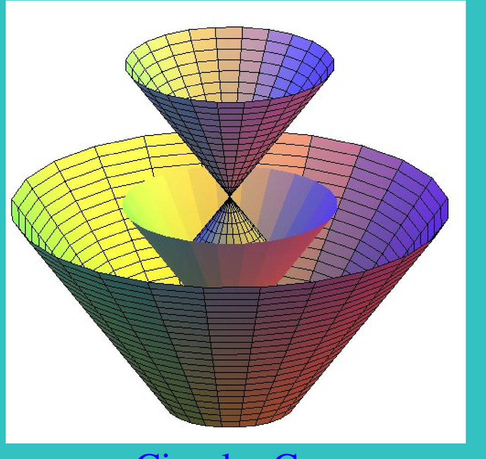
Hyperbolic Paraboloid



Two−sheeted Hyperboloid



Circular Paraboloid



Circular Cone

All the offsets shown here are rational (the cone offset has two rational componentes). For a complete analysis for the (generalized) offsets of quadrics see [8].

Rationality and Parametrization Algorithm

Let $\mathcal{P}(\bar t)$ be a rational parametrization of a surface $\mathcal{V}$ (or in general a hypersurface) and $\mathcal{N}(\bar t) = (N_1(\bar t), N_2(\bar t), N_3(\bar t))$ its associated normal vector.

Offset Rationality Characterization by means of two notions

Rational Pythagorean Hodograph (RPH)	Reparametrizing surface ($\mathcal{G}_{\mathcal{P}}(\mathcal{V})$)
$\downarrow$	$\downarrow$
$\|\mathcal{N}(\bar t)\| \in \mathbb{K}(\bar t)$	$x_3^2\sum_{i=2}^3 N_i^2 - N_2^2 - 2\,x_3\,N_1\,N_2.$

Parametrization Algorithm

GIVEN: a proper rational parametrization $\mathcal{P}(\bar t)$ of $\mathcal{V}$ in $\mathbb{K}^2$ (for curves) or $\mathbb{K}^3$ (for surfaces) and a matrix A defining a direct isometry.
DECIDE: whether the components of $\mathcal{O}_d^A(\mathcal{V})$ are rational.
DETERMINE: (in the affirmative case) a rational parametrization of each component of $\mathcal{O}_d^A(\mathcal{V})$.

1. Compute the normal vector $\mathcal{N}(\bar t)$ of $\mathcal{P}(\bar t)$
2. IF $\|\mathcal{N}\| \in \mathbb{K}(\bar t)$ THEN RETURN $\ll \mathcal{O}_d^A(\mathcal{V})$ has two rational components parametrized by $\mathcal{P} \pm \frac{d}{\|\mathcal{N}\|}\mathcal{N}\cdot A \gg$.
3. Determine $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$, and decide whether $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$ is rational. (In this situation $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$ is irreducible).
4. IF $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$ is not rational THEN RETURN $\ll \mathcal{O}_d(\mathcal{V})$ has no rational component $\gg$ ELSE
   - 4.1. Determine a rational parametrization $\mathcal{R} = (\tilde R, R)$ of $\mathcal{G}_{\mathcal{P}}(\mathcal{V})$, where $\tilde R \in \mathbb{K}(\bar t)$ or $\mathbb{K}(\bar t)^2$ depending on whether $\mathcal{V}$ is a curve or a surface, respectively.
   - 4.2. IF $\mathcal{V}$ is a plane curve THEN RETURN $\ll \mathcal{O}_d^A(\mathcal{V})$ is a rational curve parametrized by $\mathcal{Q} = \mathcal{P}(\tilde R) + \frac{2\,d\,R}{N_2(\tilde R)(R_2+1)}\mathcal{N}(\tilde R)\cdot A$, where $\mathcal{N} = (N_1, N_2)$ $\gg$.
   - 4.3. IF $\mathcal{V}$ is a surface THEN RETURN $\ll \mathcal{O}_d^A(\mathcal{V})$ is a rational surface parametrized by $\mathcal{Q} = \mathcal{P}(\tilde R) + \frac{d\,R}{(N_1(\tilde R)\,R + N_2(\tilde R))}\mathcal{N}(\tilde R)\cdot A$, where $\mathcal{N} = (N_1, N_2, N_3) \gg$.

REFERENCE FOR THESE RESULTS: [2]

Genus Formula for the Curve Case

Main difficulty: the rationality of the original variety is not preserved (in general) when the offset is considered. This phenomenon implies that the genus of the original curve is not preserved when offseting.

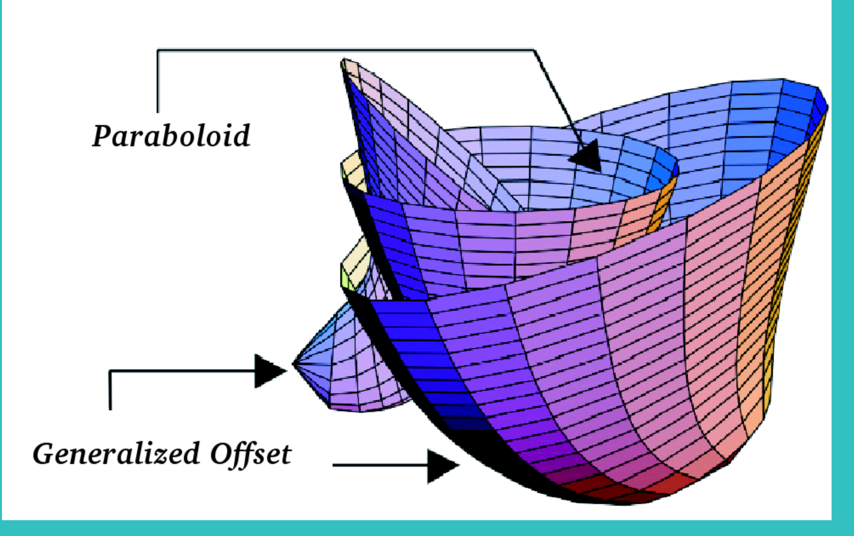
Let $\bar{\mathcal{C}}$ be an irreducible projective plane curve over $\mathbb{K}$ of degree n, satisfying that (1) all the singularities of $\bar{\mathcal{C}}$ are affine and ordinary, (2) the line at infinty $y_0 = 0$ is not tangent to $\bar{\mathcal{C}}$, (3) the curve $\bar{\mathcal{C}}$, the tangent lines to $\bar{\mathcal{C}}$ at the flex points, and the tangent lines to $\bar{\mathcal{C}}$ at the singularities do not pass through the cyclic points. Suppose that $\mathcal{O}_d^A(\mathcal{C})$ is irreducible and simple (see [9] for the notion of simple component). Let $r_1, \ldots, r_s$ be the multiplicities of its singular points. Then the generalized offset at distance d has genus

$$\boxed{g(\mathcal{O}_d^A(\mathcal{C})) = 2n^2 - 4n + 1 - 2\sum_{j=1}^{s} r_j(r_j-1).}$$

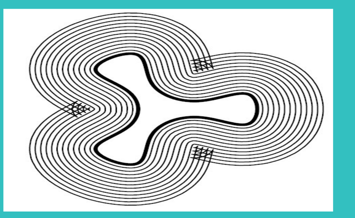
Applying this result we obtain that generalized offsets to ellipses and hyperbolas are elliptic curves.

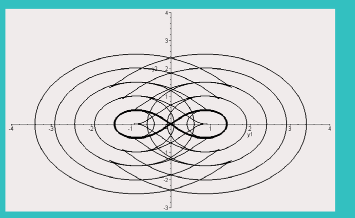
REFERENCE FOR THESE RESULTS: [3]

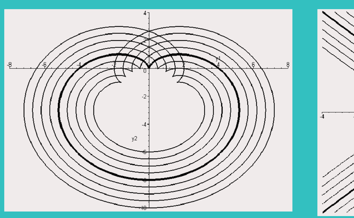
An Example of the Generalized Offset of a Surface

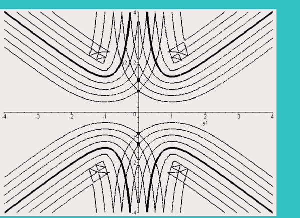


This generalized offset is also rational.









Offset Degree Formulas for Plane Curves

Implicit case

Let $\mathcal{C}$ be a real irreducible plane algebraic curve defined by the polynomial $f(y_1, y_2) \in \mathbb{R}[y_1, y_2]$. Let $F(y_1, y_2, y_3)$ be the homogenization of f wrt y_3, and let F_1, F_2 be the partial derivatives of f w.r.t. y_1 and y_2 respectively. Let $n = \deg(\mathcal{C})$.

Formula with resultants

We consider a certain *auxiliary curve* $\mathcal{S}$, defined by:

$$S(y_1, y_2, y_3) = (F_1^2 + F_2^2)(y_1 - ky_2)^2 - d^2(F_1 - kF_2)^2$$

Let $\mathrm{PP}_{\{k,d\}}$ denotes the primitive part w.r.t. $\{k, d\}$. Let $\mathcal{C}$ not be a line. Then, for $d \in \mathbb{R}\setminus\{0\}$ (or $d \in \mathbb{R}\setminus\{0, r\}$ if $\mathcal{C}$ is a circle of radius r) it holds that

$$\boxed{\deg(\mathcal{O}_d(\mathcal{C})) = \deg_{\{y_1,y_3\}}\left(\mathrm{PP}_{\{k,d\}}\left(\mathrm{Res}_{y_3}(F, S)\right)\right)}$$

This is a *deterministic* formula for the offset degree, *well suited for computation.*

Formula with the hodograph

The *hodograph curve* to $\mathcal{C}$ is the curve $\mathcal{H}$ defined by the polynomial $H(y_1, y_2, y_3) = F_1^2 + F_2^2$. Let $\mathcal{F}_\infty$ be the set of intersection points at infinity of $\overline{\mathcal{C}}$ and $\overline{\mathcal{H}}$, and let $\mathcal{F}_a$ be the affine singular locus of $\overline{\mathcal{C}}$. For $p = (a : b : 0) \in \mathcal{F}_\infty$, with $b \neq 0$, let $A_P = \min\left(\mathrm{mult}_p(\overline{\mathcal{C}}, \overline{\mathcal{H}}), \mathrm{mult}_p(\overline{\mathcal{C}}, y_3^2F_1^2)\right)$ (If $b = 0$, use $y_3^2F_2^2$ instead of $y_3^2F_1^2$). Then, for $d \in \mathbb{R}\setminus\{0\}$ (or $d \in \mathbb{R}\setminus\{0, r\}$ if $\mathcal{C}$ is a circle of radius r) it holds that:

$$\boxed{\deg(\mathcal{O}_d(\mathcal{C})) = 2\deg(\mathcal{C}) - \sum_{p\in \mathrm{Sing}_a(\overline{\mathcal{C}})} \mathrm{mult}_p(\overline{\mathcal{C}}, \overline{\mathcal{H}}) - \sum_{p\in\mathcal{F}_\infty} A_p}$$

Note: alternative works on this topic can be found in [1].

Rational case

Let $\mathcal{C}$ be a real rational plane curve with parametrization $(X(t)/W(t), Y(t)/W(t))$, where $\gcd(X, Y, W) = 1$, and let

$$N_1 = WX' - W'X,\ N_2 = WY' - W'Y,$$
$$N = N_1^2 + N_2^2,\ n = \max(\deg_t(X), \deg_t(Y), \deg_t(W)),$$
$$\tau = \max(\deg_t(N_1), \deg(N_2)),$$
$$\mu = \deg_t(\gcd(N\gcd(X, Y)^2, W^2\gcd(N_1, N_2)^2)).$$

Then, for $d \in \mathbb{R}\setminus\{0\}$ (or $d \in \mathbb{R}\setminus\{0, r\}$ if $\mathcal{C}$ is a circle of radius r) it holds that

$$\boxed{\deg(\mathcal{O}_d(\mathcal{C})) = 2n + 2\tau - \mu}$$

This provides a simpler alternative to the previous formula presented by Farouki and Neff ([5]).

REFERENCE FOR THESE RESULTS: [7]

Ongoing Work

Our next goal is to extend the degree formulae to the case of offsets of rational surfaces in three dimensional space, and, more generally, offsets of hypersurfaces. The general strategy remains the same: if Σ is a surface, we study the degree of its offset $\mathcal{O}_d(\Sigma)$ by means of the intersection with a generic line. By means of the parametrization of Σ the problem can be reduced to a plane curve intersection problem, between some analogues of of the auxiliary curve $\mathcal{S}$. This leads to a temptative degree formula for rational surfaces which only involves univariate resultants and gcds. This formula has been verified to hold in the case of quadrics, but there are still some details missing in the proof.

References

1. F. Anton, I. Emiris, B. Mourrain, and M. Teillaud (2005), *The offset to an algebraic curve and an application to conics.* O. Gervasi et al. (Eds.): ICCSA 2005, LNCS 3480, pp.683-696.
2. Arrondo E., Sendra J., Sendra J.R. (1997), *Parametric Generalized Offsets to Hypersurfaces*. J. of Symbolic Computation (vol. 23), pp. 267-285.
3. Arrondo E., Sendra J., Sendra J. R. (1999). *Genus Formula for Generalized Offset Curves*. Journal of Pure and Applied Algebra (vol. 136), no. 3, pp. 199–209.
4. Farin G., Hoschek J., Kim M.-S. (2002). *Handbook of Computer Aided Geometric Design*. North-Holland.
5. Farouki R. T. , Neff C. A. (1990). *Algebraic properties of plane offset curves*. Computer Aided Geometric Design vol. 7, pp. 101-127.
6. Pottmann H. (1995). *Rational Curves and Surfaces with Rational Offsets*. Computer Aided Geometric Design vol. 12, pp. 175–192.
7. San Segundo F., Sendra J. R. (2005). *Degree Formulae for Offset Curves*. Journal of Pure and Applied Algebra, vol.195(3), pp. 301-335.
8. Sendra J., Sendra J. R. (2000). *Rationality Analysis and Direct Parametrization of Generalized Offsets to Quadrics*. Applicable Algebra in Engineering, Communication and Computing vol. 11, no. 2, pp. 111–139.
9. Sendra J., Sendra J. R. (2000). *Algebraic Analysis of Offsets to Hypersurfaces*. Mathematische Zeitschrift vol. 234, pp. 697–719.

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