Rational Conchoids of Rational Ruled Surfaces

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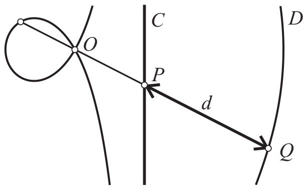
Abstract

The *conchoid* of a surface F with respect to a given fixed point O (*focus*) is roughly speaking the surface obtained by increasing the *radius function* with respect to O by a constant. We study real rational ruled surfaces and prove that their conchoids possess real rational parametrizations, independently on the focus.

Conchoid

The conchoid is a classical geometric construction and dates back already to the ancient Greeks.



The conchoid D of C with respect to O at distance d is the set of points defined by,

$$D = \{Q \in OP \text{ with } P \in C, \text{ and } \overline{QP} = d\}.$$

The definition of a *conchoid surface* to a given surface F in space with respect to a given point O and distance d follows analogous lines. For an analytic representation it is convenient to choose O as the origin. Using a representation of a surface F in terms of *polar coordinates* $\mathbf{f}(u,v) = r(u,v)\mathbf{k}(u,v)$ with $\|\mathbf{k}\| = 1$, its conchoid is obtained by

$$\mathbf{g}(u,v) = (r(u,v) \pm d)\mathbf{k}(u,v). \quad (1)$$

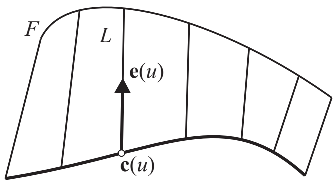
For a more algebraically definition see [2].

Ruled Surface

A *ruled surface* F carries a one-parameter family of straight lines, thus admits a parametric representation

$$\mathbf{f}(u,v) = \mathbf{c}(u) + v\,\mathbf{e}(u),$$

where $\mathbf{c}(u)$ is called *directrix curve* and $\mathbf{e}(u)$ is a *direction vector field* of F's generating lines.



Conchoids of Rational Ruled Surfaces

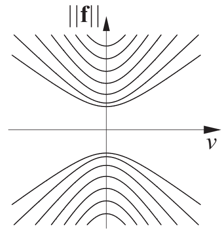
Theorem: The conchoid surface G of a rational ruled surface F in $\mathbb{R}^3$ is rational and real rational parametrizations are constructed explicitly.

Sketch of proof: Let a rational ruled surface F be given by $\mathbf{f}(u,v) = \mathbf{c}(u) + v\mathbf{e}(u)$, with $\mathbf{c}, \mathbf{e}$ rational. In order to find a rational polar representation of F with respect to the origin, we investigate the squared length $\|\mathbf{f}\|^2$ of F. With $\|\mathbf{f}\| = \frac{x_1}{x_0}$ and $v = \frac{x_2}{x_0}$ it reads

$$x_0^2\mathbf{c}^2 + 2x_0x_2\mathbf{c}\cdot\mathbf{e} + x_2^2\mathbf{e}^2 - x_1^2 = 0. \quad (2)$$

Equation (2) defines a one parameter family of conics $A(u) \in \mathbb{P}^2$.

It can be proved that there exist rational functions $x_0(u), x_1(u)$ and $x_2(u)$ which satisfy (2) identically for all u, see [3]. Choose $\mathbf{c}$ as the *footpoint curve* of F then (2) reads $Lx_0^2 + Mx_1^2 + Nx_2^2 = 0$ and so the rational curve $\mathbf{x}(u) = (x_0, x_1, x_2)(u)$ can be constructed explicitly by investigating the zeros of the polynomials $L(u), M(u)$ and $N(u)$, see [1].



For some surfaces $\|\mathbf{e}(u)\|$ is rational. Reparametrize so that $\forall u : \|\mathbf{e}(u)\| = 1$ then $A(u)$ share two common points. Stereographic projections with centers on $\mathbf{x}$ lead to a rational parametrization of (2) and thus to a rational polar parametrization of F.

Plane

The plane $F : z = 1$ is represented by the radius function $r(u,v) = 1/\sin v$ and

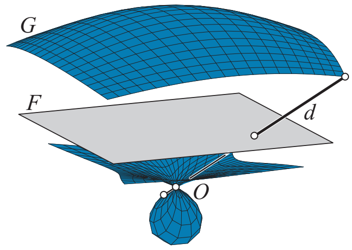
$$\mathbf{k}(u,v) = (\cos u \cos v, \sin u \cos v, \sin v).$$

Thus the conchoid surface G of the plane F admits a trigonometric parametrization

$$\mathbf{g}(u,v) = \frac{1 + d\sin v}{\sin v}\mathbf{k}(u,v),$$

and an implicit equation of G reads

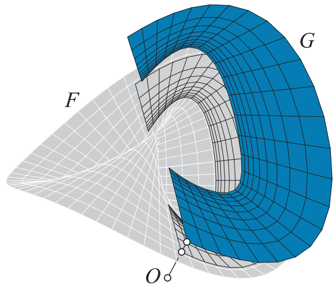
$$G : z^2(x^2+y^2+z^2) - 2z(x^2+y^2+z^2) + x^2 + y^2 + z^2(1-d^2) = 0.$$



Plücker Conoid

The Plücker conoid F can be generated in the following way. Rotate the x-axis around z and superimpose this rotation by the translation $(0,0,\sin 2u)$ in z-direction. To obtain a non trivial case we translate the surface and get a trigonometric parametrization of F,

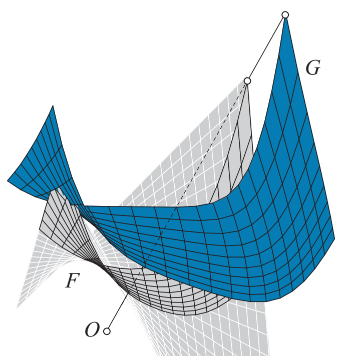
$$\mathbf{f}(u,v) = (v\cos u, v\sin u + 1, \sin 2u + 2).$$



Hyperbolic Paraboloid

A hyperbolic paraboloid F can be generated in the following way. Choose the lines $(u,1,u)$ and $(u,-1,-u)$ as directrices of F. We translate again to get a polynomial parametrization of F,

$$\mathbf{f}(u,v) = (u, v, uv+1).$$



References

[1] Peternell, M., 1997. Rational Parametrizations for Envelopes of Quadric Families, Thesis, University of Technology, Vienna.

[2] Sendra, J., and Sendra, J.R., 2010. Rational parametrization of conchoids to algebraic curves, Applicable Algebra in Engineering, Communication and Computing, Vol.21, No.4, pp. 413–428.

[3] Shafarevich, R.I., 1994. Basic Algebraic Geometry, Vol.I, Springer, Heidelberg.