Symbolic Computation of Drazin Inverses

J. Sendra
Dpto. de Matemática Aplicada a las TIC.
Universidad Politécnica de Madrid. Spain

This work was supported by the Spanish Ministerio de Economía y Competitividad, and by the European Regional Development Fund (ERDF), under the Project **MTM2014-54141-P**.

Motivation

Motivated on the notion of generalized inverse introduced by Moore and Penrose, Drazin, in [3], introduced the notion of Drazin inverse in the more general context of rings and semigroups. For matrices, Drazin inverse is defined as follows. Let A be an square matrix over a field, then the Drazin index of A is the smallest non-negative integer k such that $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$; let us denote it by $\operatorname{index}(A)$. In this situation, the Drazin inverse of A is the unique matrix satisfying the following matrix equations:

$$\begin{cases} A^{\operatorname{index}(A)+1} \cdot X = A^{\operatorname{index}(A)} \\ A \cdot X = X \cdot A \\ X \cdot A \cdot X = X. \end{cases} \qquad (1)$$

Many authors have analyzed the computation of the Drazin inverse, [1], [2]. The problem has been approached mainly for matrices with complex numbers. Nevertheless, in a second stage, different authors have addressed the problem of computing Drazin inverses of matrices over other coefficients domains as rational function fields, see [4], [5], [6]. This led us to raising the following challenge:

Problem: *We want to* compute *the Drazin inverses of matrices whose entries are elements of a finite transcendental field extension of a computable field.*

Our Results

For this purpose, we reduce the computation of Drazin inverses over certain computable fields to the computation of Drazin inverses of matrices with rational functions as entries. As a consequence, we derive a symbolic algorithm. The algorithm is applied to matrices over the field of meromorphic functions, in several complex variables, on a connected domain and to matrices over the field of Laurent formal power series. Essentially, this algorithmic method applies symbolic computation to determine the Drazin inverse via specializations, and reduces the problem to the computation, via Gröbner bases, of Drazin inverse matrices with multivariate rational functions as entries. Furthermore, we show how to relate the specialization of the Drazin inverse of a matrix, with meromorphic function entries, and the Drazin inverse of the specialization.

The results mentioned above have been developed in the papers [4] and [5].

More precisely, given a matrix A, the idea consists in the following four steps (see Fig. 2):

1. [Specialization step] first we associate to A a matrix A^* whose entries are rational functions in several variables;
2. [Inverse computation step] we compute the Drazin inverse of A^* via Gröbner bases, (see Algorithm 1 in Fig. 1);
3. [Specialization Test] we check when A specializes properly. More precisely, we proved the existence, and actual computation, of a multivariate polynomial (that we call evaluation polynomial), such that if it does not vanish it concludes that A specializes properly.
4. [Specialization of the inverse step] finally, in case of affirmative answer of the test of step 3, we get the Drazin inverse of A from the Drazin inverse of A^*.

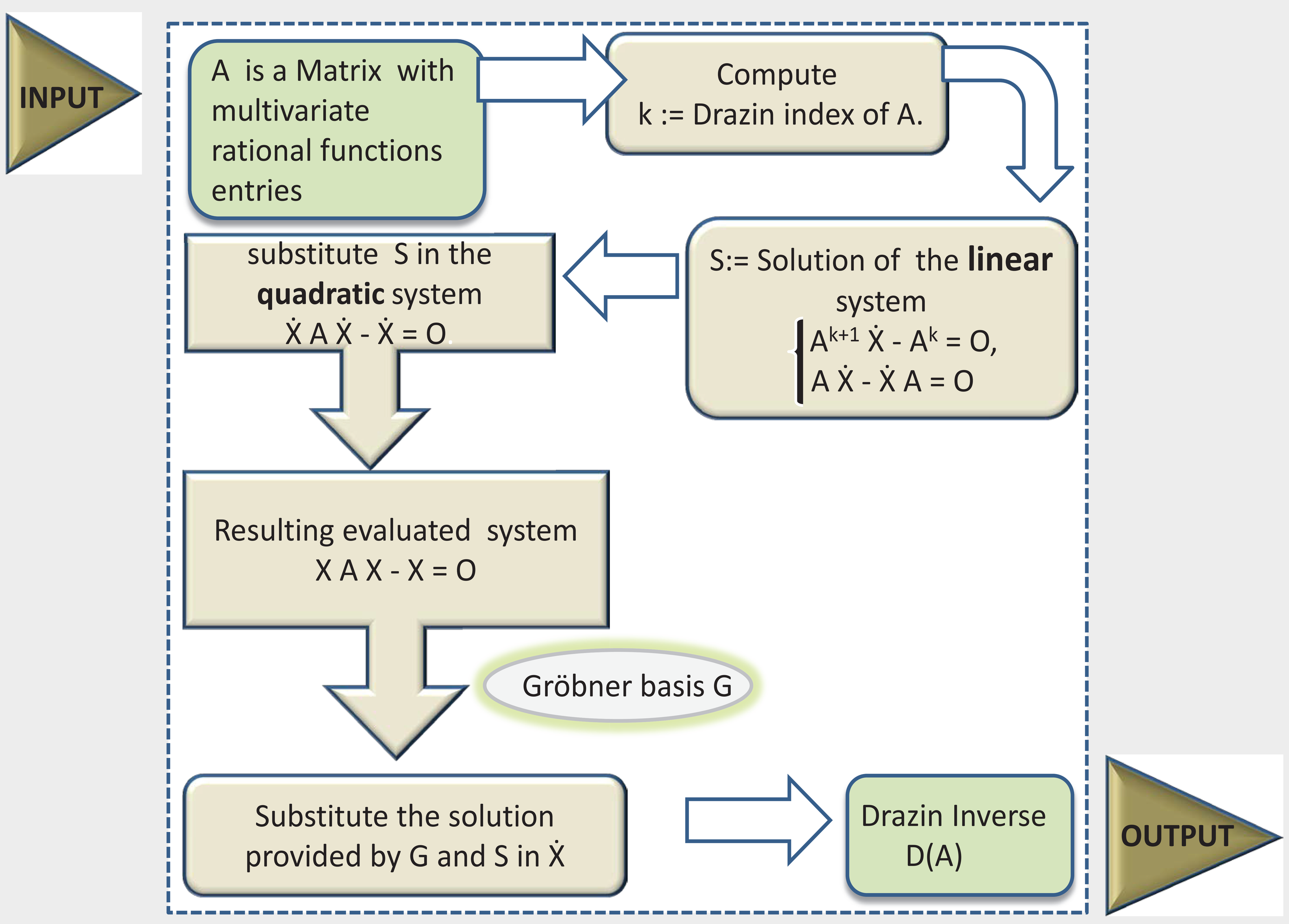

Fig.1 Scheme of Algorithm 1 in Step 2

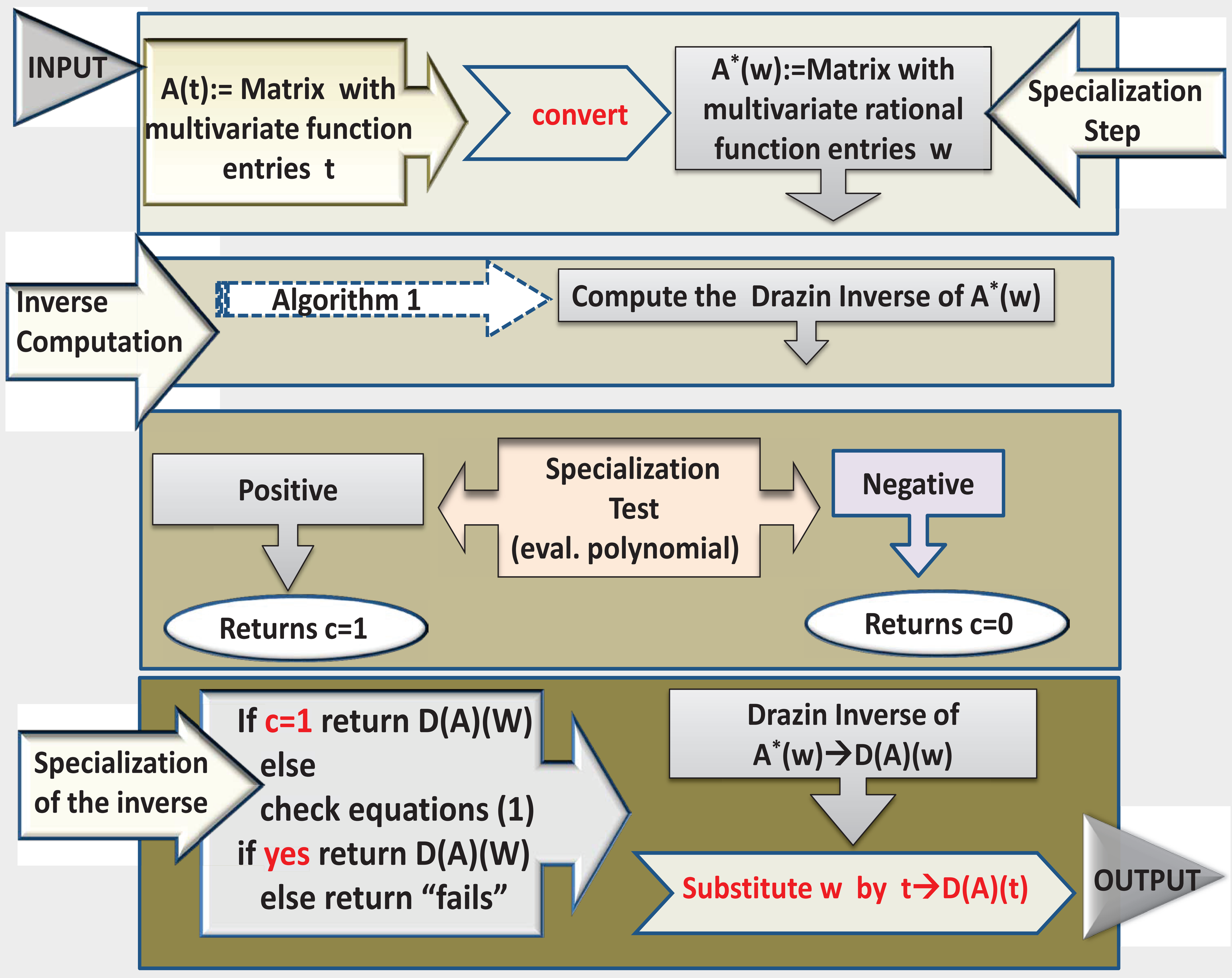

Fig.2 General Scheme

An Example

Let $\mathbf{t} = (\cos(z), e^z)$ and $\mathbf{w} = (w_1, w_2)$. Let

$$A(\mathbf{t}) = \begin{pmatrix} 0 & 0 & 2\,e^z \cos(z) \\ 2\cos(z)\,e^{-z} & 2\,e^{-z} & 2 - e^{-z} \\ 3\cos(z)\,e^{-z} & 3\,e^{-z} & 6\,e^{-z} \end{pmatrix} \in \mathcal{M}_{n\times n}(\mathbb{C}(\mathbf{t})).$$

We want to compute the Drazin inverse $\mathfrak{D}(A)$ of A.

For this purpose, we first associate to A a matrix $A^* \in \mathcal{M}_{3\times 3}(\mathbb{C}(\mathbf{w}))$.

[Specialization step]: By replacing $w_1 := \cos(z), w_2 := e^z$ we get

$$A^*(\mathbf{w}) = \begin{pmatrix} 0 & 0 & 2\,w_2 w_1 \\ \frac{2\,w_1}{w_2} & \frac{2}{w_2} & \frac{2}{w_2} \\ \frac{3\,w_1}{w_2} & \frac{3}{w_2} & \frac{6}{w_2} \end{pmatrix} \in \mathcal{M}_{n\times n}(\mathbb{C}(\mathbf{w}))$$

[Inverse computation step]: Applying Algorithm 1 in Step 2 we get

$$\mathfrak{D}(A^*)(\mathbf{w}) = \begin{pmatrix} \frac{-4\,{w_2}^3{w_1}^2}{3\,({w_1}^4{w_2}^4 - 2\,{w_1}^2{w_2}^2+1)} & \frac{-4\,{w_2}^3 w_1}{3\,({w_1}^2{w_2}^2-1)^2} & \frac{3\,({w_1}^2{w_2}^2+5)\,{w_2}^3 w_1}{9\,({w_1}^2{w_2}^2-1)^2} \\ \frac{({w_1}^2{w_2}^2+3)\,w_2 w_1}{3\,({w_1}^2{w_2}^2-1)^2} & \frac{({w_1}^2{w_2}^2+3)\,w_2}{3\,({w_1}^2{w_2}^2-1)^2} & -\frac{(5\,{w_1}^2{w_2}^2+3)\,w_2}{9\,({w_1}^2{w_2}^2-1)^2} \\ \frac{w_2 w_1}{2\,({w_1}^2{w_2}^2-1)} & \frac{w_2}{2\,({w_1}^2{w_2}^2-1)} & \frac{-w_2}{3\,({w_1}^2{w_2}^2-1)} \end{pmatrix}.$$

[Specialization Test]: we get the evaluation polynomial

$$\operatorname{EvalPol}_{A,A^*}(\mathbf{w}) = 144 w_2 w_1 (w_2 w_1 - 1)(w_1 w_2 + 1) \in \mathbb{C}[\mathbf{w}].$$

and we evaluate it at $\mathbf{t}$ to get $T((\mathbf{z})) = 144 e^{\mathbf{z}} \cos(\mathbf{z})(e^{\mathbf{z}}\cos(\mathbf{z}) - 1)(e^{\mathbf{z}}\cos(\mathbf{z}) + 1)$. Taking $z_0 = \pi$, we get that $K = T(\mathbf{t}(\pi)) = -144 e^{\pi}(e^{2\pi} - 1) \neq 0$, and hence Algorithm returns affirmative answer $c = \mathbf{1}$.

[Specialization of the inverse step]: finally, replacing **w** by **t** in the Drazin inverse of A^*, we get the Drazin inverse of A:

$$\mathfrak{D}(A^*) = \begin{pmatrix} \frac{-4\,(e^z)^3\cos(z)^2}{3\,(\cos(z)^4(e^z)^4 - 2\cos(z)^2(e^z)^2+1)} & \frac{-4\,(e^z)^3\cos(z)}{3\left(\cos(z)^2(e^z)^2-1\right)^2} & \frac{3\left(\cos(z)^2(e^z)^2+5\right)(e^z)^3\cos(z)}{9\left(\cos(z)^2(e^z)^2-1\right)^2} \\ \frac{\left(\cos(z)^2(e^z)^2+3\right)e^z\cos(z)}{3\left(\cos(z)^2(e^z)^2-1\right)^2} & \frac{\left(\cos(z)^2(e^z)^2+3\right)e^z}{3\left(\cos(z)^2(e^z)^2-1\right)^2} & \frac{-\left(5\cos(z)^2(e^z)^2+3\right)e^z}{9\left(\cos(z)^2(e^z)^2-1\right)^2} \\ \frac{e^z\cos(z)}{2\left(\cos(z)^2(e^z)^2-1\right)} & \frac{e^z}{2\left(\cos(z)^2(e^z)^2-1\right)} & \frac{-e^z}{3\left(\cos(z)^2(e^z)^2-1\right)} \end{pmatrix}.$$

References

[1] Ben-Israel A., Greville T.N.E. Generalized Inverses: Theory and Applications, second ed., Springer-Verlag, Berlin, 2003.

[2] Campbell S.L. Carl D. Meyer C.D. Generalized Inverses of Linear Transformations *Series: Classics in Applied Mathematics*. SIAM (2009)

[3] Drazin M.P. *Pseudo inverse in associative rings and semigroups*, Amer. Math. Monthly 65 (1958) 506–514.

[4] Sendra J., Sendra J.R. Gröbner Basis Computation of Drazin Inverses with Multivariate Rational Function Entries. *Applied Mathematics and Computation* **259** (2015) pp. 450-459.

[5] Sendra J.R, Sendra J. Symbolic computation of Drazin inverses by specializations *Journal of Computational and Applied Mathematics* **301** (2016) pp. 201-212.

[6] Stanimirović P.S, Milan B. Tasić M.B., Vu K.M. Extensions of Faddeevs algorithms to polynomial matrices. *Applied Mathematics and Computation* **214** (2009) 246–258.